Kodaira embedding and a Schwarz Lemma after Yau-Royden

Lei Ni

University of California, San Diego

September, 2019

Outline

Outline

- A. Kodaira's embedding and its implications.

Outline

- A. Kodaira's embedding and its implications.
- B. Two new curvature notions and intrinsic characterizations of the projectivity/algebraicity.

Outline

- A. Kodaira's embedding and its implications.
- B. Two new curvature notions and intrinsic characterizations of the projectivity/algebraicity.
- C. Two new theorems and the proof.

Outline

- A. Kodaira's embedding and its implications.
- B. Two new curvature notions and intrinsic characterizations of the projectivity/algebraicity.
- C. Two new theorems and the proof.
- D. Yau-Royden's Schwarz Lemma and Kobayashi (and k-) hyperbolicity.

Outline

- A. Kodaira's embedding and its implications.
- B. Two new curvature notions and intrinsic characterizations of the projectivity/algebraicity.
- C. Two new theorems and the proof.
- D. Yau-Royden's Schwarz Lemma and Kobayashi (and k-) hyperbolicity.
- E. A new Schwarz Lemma and implications.

Outline

- A. Kodaira's embedding and its implications.
- B. Two new curvature notions and intrinsic characterizations of the projectivity/algebraicity.
- C. Two new theorems and the proof.
- D. Yau-Royden's Schwarz Lemma and Kobayashi (and k-) hyperbolicity.
- E. A new Schwarz Lemma and implications.
- F. Conjectures, Kähler C-spaces, CQB, CQB^d, and some progresses.

Outline

- A. Kodaira's embedding and its implications.
- B. Two new curvature notions and intrinsic characterizations of the projectivity/algebraicity.
- C. Two new theorems and the proof.
- D. Yau-Royden's Schwarz Lemma and Kobayashi (and k-) hyperbolicity.
- E. A new Schwarz Lemma and implications.
- F. Conjectures, Kähler C-spaces, CQB, CQB^d, and some progresses.

  Some of results are joint work with F. Zheng.

Outline

- A. Kodaira's embedding and its implications.
- B. Two new curvature notions and intrinsic characterizations of the projectivity/algebraicity.
- C. Two new theorems and the proof.
- D. Yau-Royden's Schwarz Lemma and Kobayashi (and k-) hyperbolicity.
- E. A new Schwarz Lemma and implications.
- F. Conjectures, Kähler C-spaces, CQB, CQB^d, and some progresses.

  Some of results are joint work with F. Zheng.

  Papers can be downloaded at
  http://math.ucsd.edu/ ~ lni/academic.html

A1. Kodaira embedding

A1. Kodaira embedding

A1. Kodaira embedding

- Embedding problem/theorem is universal. It unites the extrinsic and intrinsic geometry.

A1. Kodaira embedding

- Embedding problem/theorem is universal. It unites the extrinsic and intrinsic geometry.
- Whitney (1936): any smooth real n-dimensional manifold (Hausdorff and second-countable) can be smoothly embedded in the real $\mathbb{R}^{2n}$, $n \geq 1$ (weak version in $\mathbb{R}^{2n+1}$).

A1. Kodaira embedding

- Embedding problem/theorem is universal. It unites the extrinsic and intrinsic geometry.
- Whitney (1936): any smooth real n-dimensional manifold (Hausdorff and second-countable) can be smoothly embedded in the real $\mathbb{R}^{2n}$, $n \geq 1$ (weak version in $\mathbb{R}^{2n+1}$).
- Nash(1956): any C^∞ Riemannian manifold (M^n, g) can be isometrically embedded into some Euclidean space $\mathbb{R}^d$ with $d = d(n)$. Nash (1951): algebraic embedding of real manifolds

A1. Kodaira embedding

- Embedding problem/theorem is universal. It unites the extrinsic and intrinsic geometry.
- Whitney (1936): any smooth real n-dimensional manifold (Hausdorff and second-countable) can be smoothly embedded in the real $\mathbb{R}^{2n}$, $n \geq 1$ (weak version in $\mathbb{R}^{2n+1}$).
- Nash(1956): any C^∞ Riemannian manifold (M^n, g) can be isometrically embedded into some Euclidean space $\mathbb{R}^d$ with $d = d(n)$. Nash (1951): algebraic embedding of real manifolds
- Remmert(1956)-Narasimhan(1960)-Bishop(1961): every Stein manifold M^m can be holomorphically embedded into $\mathbb{C}^{2m+1}$.

A1. Kodaira embedding

- Embedding problem/theorem is universal. It unites the extrinsic and intrinsic geometry.
- Whitney (1936): any smooth real n-dimensional manifold (Hausdorff and second-countable) can be smoothly embedded in the real $\mathbb{R}^{2n}$, $n \geq 1$ (weak version in $\mathbb{R}^{2n+1}$).
- Nash(1956): any C^∞ Riemannian manifold (M^n, g) can be isometrically embedded into some Euclidean space $\mathbb{R}^d$ with $d = d(n)$. Nash (1951): algebraic embedding of real manifolds
- Remmert(1956)-Narasimhan(1960)-Bishop(1961): every Stein manifold M^m can be holomorphically embedded into $\mathbb{C}^{2m+1}$.

A1. Kodaira embedding

- Embedding problem/theorem is universal. It unites the extrinsic and intrinsic geometry.
- Whitney (1936): any smooth real n-dimensional manifold (Hausdorff and second-countable) can be smoothly embedded in the real $\mathbb{R}^{2n}$, $n \geq 1$ (weak version in $\mathbb{R}^{2n+1}$).
- Nash(1956): any C^∞ Riemannian manifold (M^n, g) can be isometrically embedded into some Euclidean space $\mathbb{R}^d$ with $d = d(n)$. Nash (1951): algebraic embedding of real manifolds
- Remmert(1956)-Narasimhan(1960)-Bishop(1961): every Stein manifold M^m can be holomorphically embedded into $\mathbb{C}^{2m+1}$.
- A Kähler manifold (M, g) is a complex one whose Kähler form $\omega_g = \frac{\sqrt{-1}}{2} g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$ is d-closed.

A1. Kodaira embedding

- Embedding problem/theorem is universal. It unites the extrinsic and intrinsic geometry.
- Whitney (1936): any smooth real n-dimensional manifold (Hausdorff and second-countable) can be smoothly embedded in the real $\mathbb{R}^{2n}$, $n \geq 1$ (weak version in $\mathbb{R}^{2n+1}$).
- Nash(1956): any C^∞ Riemannian manifold (M^n, g) can be isometrically embedded into some Euclidean space $\mathbb{R}^d$ with $d = d(n)$. Nash (1951): algebraic embedding of real manifolds
- Remmert(1956)-Narasimhan(1960)-Bishop(1961): every Stein manifold M^m can be holomorphically embedded into $\mathbb{C}^{2m+1}$.
- A Kähler manifold (M, g) is a complex one whose Kähler form $\omega_g = \frac{\sqrt{-1}}{2} g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$ is d-closed.
- Kodaira (1954): Assuming the Kähler form is integral, a compact Kähler manifold can be holomorphically embedded into the complex projective $\mathbb{P}^m$ (called projective).

A2. Implications

- Any projective manifold is Kähler (complex submanifolds of Kähler are Kähler). Not all Kähler manifolds are projective.

A2. Implications

- Any projective manifold is Kähler (complex submanifolds of Kähler are Kähler). Not all Kähler manifolds are projective.
- 'Forgetting' is helpful: Serre's duality for algebraic manifolds can be proved by Hodge-Kodaira theorem on complex vector bundles of any complex manifolds.

A2. Implications

- Any projective manifold is Kähler (complex submanifolds of Kähler are Kähler). Not all Kähler manifolds are projective.
- 'Forgetting' is helpful: Serre's duality for algebraic manifolds can be proved by Hodge-Kodaira theorem on complex vector bundles of any complex manifolds.
- W.-L. Chow's theorem (any complex submanifolds of $\mathbb{P}^m$ must be algebraic, namely zeros of polynomials) and Serre's GAGA principle.

A2. Implications

- Any projective manifold is Kähler (complex submanifolds of Kähler are Kähler). Not all Kähler manifolds are projective.
- 'Forgetting' is helpful: Serre's duality for algebraic manifolds can be proved by Hodge-Kodaira theorem on complex vector bundles of any complex manifolds.
- W.-L. Chow's theorem (any complex submanifolds of $\mathbb{P}^m$ must be algebraic, namely zeros of polynomials) and Serre's GAGA principle.
- Hence 'projectivity'='being algebraic'; and the embedding avails us the algebraic tools to study some Kähler manifolds.

A2. Implications

- Any projective manifold is Kähler (complex submanifolds of Kähler are Kähler). Not all Kähler manifolds are projective.
- 'Forgetting' is helpful: Serre's duality for algebraic manifolds can be proved by Hodge-Kodaira theorem on complex vector bundles of any complex manifolds.
- W.-L. Chow's theorem (any complex submanifolds of $\mathbb{P}^m$ must be algebraic, namely zeros of polynomials) and Serre's GAGA principle.
- Hence 'projectivity'='being algebraic'; and the embedding avails us the algebraic tools to study some Kähler manifolds.
- 'Forgetting' allows PDEs/geometric method to study the algebraic manifolds, e.g. yielding the Hodge theorem/structure, Rieman-Roch-Hirzebruch index theorem etc.

A3. The positivity of line bundles

A3. The positivity of line bundles

- An equivalent formulation of the condition for Kodaira's embedding theorem is that M admits a positive holomorphic line bundle (by also Kodaira).

A3. The positivity of line bundles

- An equivalent formulation of the condition for Kodaira's embedding theorem is that M admits a positive holomorphic line bundle (by also Kodaira).

- The positivity of L = there exists a hermitian metric a on L such that its Chern curvature $c_1(L) = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log a > 0$.

A3. The positivity of line bundles

- An equivalent formulation of the condition for Kodaira's embedding theorem is that M admits a positive holomorphic line bundle (by also Kodaira).
- The positivity of L = there exists a hermitian metric a on L such that its Chern curvature $c_1(L) = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log a > 0$.
- Kodaira's embedding theorem is built upon his vanishing theorem, which is very influential to L^2-estimate of $\bar{\partial}$-operator.

A3. The positivity of line bundles

- An equivalent formulation of the condition for Kodaira's embedding theorem is that M admits a positive holomorphic line bundle (by also Kodaira).
- The positivity of $L =$ there exists a hermitian metric a on L such that its Chern curvature $c_1(L) = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log a > 0$.
- Kodaira's embedding theorem is built upon his vanishing theorem, which is very influential to L^2-estimate of $\bar{\partial}$-operator.
- Given a Kähler manifold (M, g), when it admits a positive line bundle?

A3. The positivity of line bundles

- An equivalent formulation of the condition for Kodaira's embedding theorem is that M admits a positive holomorphic line bundle (by also Kodaira).
- The positivity of L = there exists a hermitian metric a on L such that its Chern curvature $c_1(L) = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log a > 0$.
- Kodaira's embedding theorem is built upon his vanishing theorem, which is very influential to L^2-estimate of $\bar{\partial}$-operator.
- Given a Kähler manifold (M, g), when it admits a positive line bundle?
- The 'canonical choice' is the canonical line bundle $(K_M, \det(g)^{-1})$ or anti-canonical line $K_M^{-1} = \det T'M$ bundle.

B1. Intrinsic positivities

B1. Intrinsic positivities

- For K_M, the positivity amounts to $Ric < 0$; for K_M^{-1} the positivity amounts to $Ric > 0$ (called Fano).

B1. Intrinsic positivities

- For K_M, the positivity amounts to $Ric < 0$; for K_M^{-1} the positivity amounts to $Ric > 0$ (called Fano).
- Are there other intrinsic curvatures for Kähler manifolds imply projectivity?

B1. Intrinsic positivities

- For K_M, the positivity amounts to $Ric < 0$; for K_M^{-1} the positivity amounts to $Ric > 0$ (called Fano).
- Are there other intrinsic curvatures for Kähler manifolds imply projectivity?
- Sectional curvature and bisectional curvature ($B = R(X, \bar{X}, Y, \bar{Y})$) positivity imply $Ric > 0$.

B1. Intrinsic positivities

- For K_M, the positivity amounts to $Ric < 0$; for K_M^{-1} the positivity amounts to $Ric > 0$ (called Fano).
- Are there other intrinsic curvatures for Kähler manifolds imply projectivity?
- Sectional curvature and bisectional curvature $(B = R(X, \bar{X}, Y, \bar{Y}))$ positivity imply $Ric > 0$.
- They are very strong conditions for Kähler manifolds.

B1. Intrinsic positivities

- For K_M, the positivity amounts to $Ric < 0$; for K_M^{-1} the positivity amounts to $Ric > 0$ (called Fano).
- Are there other intrinsic curvatures for Kähler manifolds imply projectivity?
- Sectional curvature and bisectional curvature ($B = R(X, \bar{X}, Y, \bar{Y})$) positivity imply $Ric > 0$.
- They are very strong conditions for Kähler manifolds.
- Mori, Siu-Yau: If a Kähler manifold (M, g) has $B > 0$, then $M = \mathbb{P}^m$ (biholomorphically).

B1. Intrinsic positivities

- For K_M, the positivity amounts to $Ric < 0$; for K_M^{-1} the positivity amounts to $Ric > 0$ (called Fano).
- Are there other intrinsic curvatures for Kähler manifolds imply projectivity?
- Sectional curvature and bisectional curvature ($B = R(X, \bar{X}, Y, \bar{Y})$) positivity imply $Ric > 0$.
- They are very strong conditions for Kähler manifolds.
- Mori, Siu-Yau: If a Kähler manifold (M, g) has $B > 0$, then $M = \mathbb{P}^m$ (biholomorphically).
- Mok: If a Kähler manifold (M, g) satisfies $B \geq 0$, $b_2 = 1$, then M is a compact HSS.

B2. Other intrinsic curvatures

B2. Other intrinsic curvatures

- Holomorphic sectional curvature $H(X) = R(X, \overline{X}, X, \overline{X})$ has been known to be independent of Ricci curvature (by examples Hitchin and others).

B2. Other intrinsic curvatures

- Holomorphic sectional curvature $H(X) = R(X, \overline{X}, X, \overline{X})$ has been known to be independent of Ricci curvature (by examples Hitchin and others).
- Does $H > 0$ (or $H < 0$) imply the projectivity (namely M can be embedded into $\mathbb{P}^k$)?

B2. Other intrinsic curvatures

- Holomorphic sectional curvature $H(X) = R(X, \overline{X}, X, \overline{X})$ has been known to be independent of Ricci curvature (by examples Hitchin and others).
- Does $H > 0$ (or $H < 0$) imply the projectivity (namely M can be embedded into $\mathbb{P}^k$)?
- Yes. For $H > 0$ it is a theorem of X. Yang (Cambridge J. Math. 2018). For $H < 0$ it is the combined effort of Wu-Yau (Invent. Math. 2016) and Tossati-Yang (JDG, 2017).

B2. Other intrinsic curvatures

- Holomorphic sectional curvature $H(X) = R(X, \overline{X}, X, \overline{X})$ has been known to be independent of Ricci curvature (by examples Hitchin and others).
- Does $H > 0$ (or $H < 0$) imply the projectivity (namely M can be embedded into $\mathbb{P}^k$)?
- Yes. For $H > 0$ it is a theorem of X. Yang (Cambridge J. Math. 2018). For $H < 0$ it is the combined effort of Wu-Yau (Invent. Math. 2016) and Tossati-Yang (JDG, 2017).
- The first result does not apply to Riemann surfaces of positive genus. The second result is also restrictive in high dimension as explained later.

B2. Other intrinsic curvatures

- Holomorphic sectional curvature $H(X) = R(X, \overline{X}, X, \overline{X})$ has been known to be independent of Ricci curvature (by examples Hitchin and others).
- Does $H > 0$ (or $H < 0$) imply the projectivity (namely M can be embedded into $\mathbb{P}^k$)?
- Yes. For $H > 0$ it is a theorem of X. Yang (Cambridge J. Math. 2018). For $H < 0$ it is the combined effort of Wu-Yau (Invent. Math. 2016) and Tossati-Yang (JDG, 2017).
- The first result does not apply to Riemann surfaces of positive genus. The second result is also restrictive in high dimension as explained later.
- More importantly there are many tori of complex dimension 2 which is not projective.

B2. Other intrinsic curvatures

- Holomorphic sectional curvature $H(X) = R(X, \overline{X}, X, \overline{X})$ has been known to be independent of Ricci curvature (by examples Hitchin and others).
- Does $H > 0$ (or $H < 0$) imply the projectivity (namely M can be embedded into $\mathbb{P}^k$)?
- Yes. For $H > 0$ it is a theorem of X. Yang (Cambridge J. Math. 2018). For $H < 0$ it is the combined effort of Wu-Yau (Invent. Math. 2016) and Tossati-Yang (JDG, 2017).
- The first result does not apply to Riemann surfaces of positive genus. The second result is also restrictive in high dimension as explained later.
- More importantly there are many tori of complex dimension 2 which is not projective.
- Our first result/curvature captures, to some degree, the essential connection between the intrinsic positivity and the projectivity.

B3. The k-scalar curvature

B3. The k-scalar curvature

- For $x \in M$ and $\Sigma \subset T'_x M$ a k-dimensional subspace, define

$$S_k(x, \Sigma^k) \doteqdot \frac{k(k+1)}{2Vol(\mathbb{S}^{2k-1})} \int_{|Z|=1, Z\in\Sigma} H(Z)\, d\theta(Z).$$

B3. The k-scalar curvature

- For $x \in M$ and $\Sigma \subset T'_x M$ a k-dimensional subspace, define

$$S_k(x, \Sigma^k) \doteqdot \frac{k(k+1)}{2Vol(\mathbb{S}^{2k-1})} \int_{|Z|=1, Z\in\Sigma} H(Z)\, d\theta(Z).$$

- Recall that the scalar curvature $S(x) = \sum_{i,j=1}^{m} R(E_i, \overline{E}_i, E_j, \overline{E}_j)$, where $\{E_i\}$ is a unitary basis of $T'_x M$. Berger proved that

$$S(x) = \frac{m(m+1)}{2Vol(\mathbb{S}^{2m-1})} \int_{|Z|=1, Z\in T'_x M} H(Z)\, d\theta(Z),$$

which implies $S > 0$ if $H > 0$.

B3. The k-scalar curvature

- For $x \in M$ and $\Sigma \subset T'_x M$ a k-dimensional subspace, define

$$S_k(x, \Sigma^k) \doteqdot \frac{k(k+1)}{2Vol(\mathbb{S}^{2k-1})} \int_{|Z|=1, Z \in \Sigma} H(Z)\, d\theta(Z).$$

- Recall that the scalar curvature $S(x) = \sum_{i,j=1}^{m} R(E_i, \overline{E}_i, E_j, \overline{E}_j)$, where $\{E_i\}$ is a unitary basis of $T'_x M$. Berger proved that

$$S(x) = \frac{m(m+1)}{2Vol(\mathbb{S}^{2m-1})} \int_{|Z|=1, Z \in T'_x M} H(Z)\, d\theta(Z),$$

which implies $S > 0$ if $H > 0$.

- We say $S_k(x) \geq \lambda$ ($\leq \lambda$) if $S_k(x, \Sigma) \geq \lambda$ ($S_k(x, \Sigma) \leq \lambda$) for all k-dimensional subspace $\Sigma \subset T'_x M$.

B3. The k-scalar curvature

- For $x \in M$ and $\Sigma \subset T'_x M$ a k-dimensional subspace, define

$$S_k(x, \Sigma^k) \doteqdot \frac{k(k+1)}{2Vol(\mathbb{S}^{2k-1})} \int_{|Z|=1, Z \in \Sigma} H(Z)\, d\theta(Z).$$

- Recall that the scalar curvature $S(x) = \sum_{i,j=1}^{m} R(E_i, \overline{E}_i, E_j, \overline{E}_j)$, where $\{E_i\}$ is a unitary basis of $T'_x M$. Berger proved that

$$S(x) = \frac{m(m+1)}{2Vol(\mathbb{S}^{2m-1})} \int_{|Z|=1, Z \in T'_x M} H(Z)\, d\theta(Z),$$

which implies $S > 0$ if $H > 0$.

- We say $S_k(x) \geq \lambda$ ($\leq \lambda$) if $S_k(x, \Sigma) \geq \lambda$ ($S_k(x, \Sigma) \leq \lambda$) for all k-dimensional subspace $\Sigma \subset T'_x M$.
Clearly for $k = 1$, $S_1(x) \geq \lambda$ is the same as $H(X) \geq \lambda |X|^4$, and $S_m(x) = S(x)$.

C.1 A Kodaira embedding theorem

C.1 A Kodaira embedding theorem

- Theorem (N-Zheng, 2018)

  Any compact Kähler manifold M^m with positive 2nd-scalar curvature must be projective. In fact $h^{p,0}(M) = 0$ for any $2 \leq p \leq m$.

C.1 A Kodaira embedding theorem

- Theorem (N-Zheng, 2018)
  Any compact Kähler manifold M^m with positive 2nd-scalar curvature must be projective. In fact $h^{p,0}(M) = 0$ for any $2 \leq p \leq m$.
  - It was a theorem of Kodaira asserting that $h^{2,0} = 0$ is sufficient for the projective embedding.

C.1 A Kodaira embedding theorem

- Theorem (N-Zheng, 2018)
  Any compact Kähler manifold M^m with positive 2nd-scalar curvature must be projective. In fact $h^{p,0}(M) = 0$ for any $2 \leq p \leq m$.
  - It was a theorem of Kodaira asserting that $h^{2,0} = 0$ is sufficient for the projective embedding.
  - This provides a general criterion on the projectivity. It is sharp since generic 2-tori are non-Abelian (not projective). The vanishing result also holds for $H^0(M, \Omega^{\otimes p})$. The vanishing theorem for $h^{p,0}$ with $p \geq k$ holds under $S_k > 0$ for $k \geq 3$. But no embedding theorem could be possible.

C.1 A Kodaira embedding theorem

- Theorem (N-Zheng, 2018)

  Any compact Kähler manifold M^m with positive 2nd-scalar curvature must be projective. In fact $h^{p,0}(M) = 0$ for any $2 \leq p \leq m$.

  - It was a theorem of Kodaira asserting that $h^{2,0} = 0$ is sufficient for the projective embedding.
  - This provides a general criterion on the projectivity. It is sharp since generic 2-tori are non-Abelian (not projective). The vanishing result also holds for $H^0(M, \Omega^{\otimes p})$. The vanishing theorem for $h^{p,0}$ with $p \geq k$ holds under $S_k > 0$ for $k \geq 3$. But no embedding theorem could be possible.
  - By Kodaira-Spencer, there exists smooth deformations of Kähler metrics among a family of holomorphic deformations. Hence the result also implies the stability of the projectivity for such manifolds.

C2. The proof

C2. The proof

- First ingredient: a $\partial\bar{\partial}$-lemma.

$$\partial\overline{\partial}|s|^2 = \langle \nabla s, \overline{\nabla s} \rangle - \widetilde{R}(s, \bar{s}, \cdot, \cdot)$$

where $\widetilde{R}$ stands for the curvature of the Hermitian bundle $\bigwedge^p \Omega$, and $\Omega = (T'M)^*$ is the holomorphic cotangent bundle of M. The metric on $\bigwedge^p \Omega$ is derived from the metric of M^m.

C2. The proof

- First ingredient: a $\partial\bar{\partial}$-lemma.

$$\partial\bar{\partial}|s|^2 = \langle \nabla s, \overline{\nabla s} \rangle - \widetilde{R}(s, \bar{s}, \cdot, \cdot)$$

  where $\widetilde{R}$ stands for the curvature of the Hermitian bundle $\bigwedge^p \Omega$, and $\Omega = (T'M)^*$ is the holomorphic cotangent bundle of M. The metric on $\bigwedge^p \Omega$ is derived from the metric of M^m.

- Second ingredient: 2nd variation consideration (on the minimal 2-subspaces, or k-spaces) is the key (this is motivated by Wiliking's proof of invariant conditions for Ricci flow). To prove the theorem we apply the maximum principle at x_0, where $|s|^2$ attains its maximum (s being a holomorphic $(2, 0)$-form).

C2. The proof

- First ingredient: a $\partial\bar{\partial}$-lemma.

$$\partial\overline{\partial}|s|^2 = \langle \nabla s, \overline{\nabla s}\rangle - \widetilde{R}(s, \bar{s}, \cdot, \cdot)$$

  where $\widetilde{R}$ stands for the curvature of the Hermitian bundle $\bigwedge^p \Omega$, and $\Omega = (T'M)^*$ is the holomorphic cotangent bundle of M. The metric on $\bigwedge^p \Omega$ is derived from the metric of M^m.

- Second ingredient: 2nd variation consideration (on the minimal 2-subspaces, or k-spaces) is the key (this is motivated by Wiliking's proof of invariant conditions for Ricci flow). To prove the theorem we apply the maximum principle at x_0, where $|s|^2$ attains its maximum (s being a holomorphic $(2, 0)$-form).

- In view of the compactness of the Grassmannians we find a complex two plane Σ in $T'_{x_0}M$ such that $S_2(x_0, \Sigma) = \inf_{\Sigma'} S_2(x_0, \Sigma') > 0$.

C3. Estimates needed

C3. Estimates needed

- In the following we denote $\fint f(Z)$ to be the average of the integral of the function f over $\mathbb{S}^3 \subset \Sigma$.

C3. Estimates needed

- In the following we denote $\fint f(Z)$ to be the average of the integral of the function f over $\mathbb{S}^3 \subset \Sigma$.
- For any $E \in \Sigma$ and $E' \perp \Sigma$ with $|E| = |E'| = 1$, we have that

$$
\begin{aligned}
\fint R(E, \overline{E}', Z, \overline{Z}) d\theta(Z) = \fint R(E', \overline{E}, Z, \bar{Z}) d\theta(Z) &= 0, \\
\fint R(E, \overline{E}, Z, \overline{Z}) + R(E', \overline{E}', Z, \overline{Z})\, d\theta(Z) &\geq \frac{1}{6} S_2(x_0, \Sigma), \\
\fint R(E', \overline{E}', Z, \overline{Z})\, d\theta(Z) &\geq \frac{1}{6} S_2(x_0, \Sigma).
\end{aligned}
$$

C3. Estimates needed

- In the following we denote $\fint f(Z)$ to be the average of the integral of the function f over $\mathbb{S}^3 \subset \Sigma$.
- For any $E \in \Sigma$ and $E' \perp \Sigma$ with $|E| = |E'| = 1$, we have that

$$\begin{aligned} \fint R(E, \overline{E}', Z, \overline{Z}) d\theta(Z) = \fint R(E', \overline{E}, Z, \bar{Z}) d\theta(Z) &= 0, \\ \fint R(E, \overline{E}, Z, \overline{Z}) + R(E', \overline{E}', Z, \overline{Z})\, d\theta(Z) &\geq \frac{1}{6} S_2(x_0, \Sigma), \\ \fint R(E', \overline{E}', Z, \overline{Z})\, d\theta(Z) &\geq \frac{1}{6} S_2(x_0, \Sigma). \end{aligned}$$

- The novelty of our proof is to use the minimality of Σ to get useful estimates first, and then apply them to tracing $\partial\bar{\partial}$-Lemma over Σ (only). B. Andrews (then adapted by Brendle and others) applied a similar trick to the diagonal manifolds (e.g. half of the dimension of the product manifold).

C4. Another family of intrinsic conditions

C4. Another family of intrinsic conditions

- Define $Ric(x, \Sigma)$ as the Ricci curvature of the curvature tensor restricted to the k-dimensional subspace $\Sigma \subset T'_x M$. Namely $Ric(x, \Sigma)(v, \bar{v}) = \sum_{i=1}^{k} R(E_i, \overline{E}_i, v, \bar{v})$ with $\{E_i\}$ being a unitary basis of Σ. We say that $Ric_k(x) > 0$ if $Ric(x, \Sigma) > 0$ for every k-dimensional subspace Σ.

C4. Another family of intrinsic conditions

- Define $Ric(x, \Sigma)$ as the Ricci curvature of the curvature tensor restricted to the k-dimensional subspace $\Sigma \subset T'_x M$. Namely $Ric(x, \Sigma)(v, \bar{v}) = \sum_{i=1}^{k} R(E_i, \overline{E}_i, v, \bar{v})$ with $\{E_i\}$ being a unitary basis of Σ. We say that $Ric_k(x) > 0$ if $Ric(x, \Sigma) > 0$ for every k-dimensional subspace Σ.
- Clearly when $k = 1$, Ric_k is just the holomorphic sectional curvature. And if $k = m$ Ric_k is just the Ricci curvature.

C4. Another family of intrinsic conditions

- Define $Ric(x, \Sigma)$ as the Ricci curvature of the curvature tensor restricted to the k-dimensional subspace $\Sigma \subset T'_x M$. Namely $Ric(x, \Sigma)(v, \bar{v}) = \sum_{i=1}^{k} R(E_i, \overline{E}_i, v, \bar{v})$ with $\{E_i\}$ being a unitary basis of Σ. We say that $Ric_k(x) > 0$ if $Ric(x, \Sigma) > 0$ for every k-dimensional subspace Σ.
- Clearly when $k = 1$, Ric_k is just the holomorphic sectional curvature. And if $k = m$ Ric_k is just the Ricci curvature.

- Theorem (Ni, 2019)

  Let (N^n, h) be a compact Kähler manifold with $Ric_k > 0$, for some $1 \leq k \leq n$. Then N is projective and rationally connected. In particular, $\pi_1(N) = \{0\}$.

C4. Another family of intrinsic conditions

- Define $Ric(x, \Sigma)$ as the Ricci curvature of the curvature tensor restricted to the k-dimensional subspace $\Sigma \subset T'_x M$. Namely $Ric(x, \Sigma)(v, \bar{v}) = \sum_{i=1}^{k} R(E_i, \overline{E}_i, v, \bar{v})$ with $\{E_i\}$ being a unitary basis of Σ. We say that $Ric_k(x) > 0$ if $Ric(x, \Sigma) > 0$ for every k-dimensional subspace Σ.
- Clearly when $k = 1$, Ric_k is just the holomorphic sectional curvature. And if $k = m$ Ric_k is just the Ricci curvature.

- Theorem (Ni, 2019)

  Let (N^n, h) be a compact Kähler manifold with $Ric_k > 0$, for some $1 \leq k \leq n$. Then N is projective and rationally connected. In particular, $\pi_1(N) = \{0\}$.

  - The rational connectivity was due to Heier-Wong (2015)for $k = 1$. For $k = m$, the rational connectivity was proved by Campana, Kollár-Miyaoka-Mori (90s), independently.

C4. Another family of intrinsic conditions

- Define $Ric(x, \Sigma)$ as the Ricci curvature of the curvature tensor restricted to the k-dimensional subspace $\Sigma \subset T'_x M$. Namely $Ric(x, \Sigma)(v, \bar{v}) = \sum_{i=1}^{k} R(E_i, \overline{E}_i, v, \bar{v})$ with $\{E_i\}$ being a unitary basis of Σ. We say that $Ric_k(x) > 0$ if $Ric(x, \Sigma) > 0$ for every k-dimensional subspace Σ.
- Clearly when $k = 1$, Ric_k is just the holomorphic sectional curvature. And if $k = m$ Ric_k is just the Ricci curvature.

- Theorem (Ni, 2019)

  Let (N^n, h) be a compact Kähler manifold with $Ric_k > 0$, for some $1 \leq k \leq n$. Then N is projective and rationally connected. In particular, $\pi_1(N) = \{0\}$.

  - The rational connectivity was due to Heier-Wong (2015)for $k = 1$. For $k = m$, the rational connectivity was proved by Campana, Kollár-Miyaoka-Mori (90s), independently.
  - The proof uses 1) the above estimates and considerations, 2) Applying maximum principle to the co-mass of the forms.

D1. Kobayashi hyperbolicity and the Schwarz Lemma of Yau-Royden

D1. Kobayashi hyperbolicity and the Schwarz Lemma of Yau-Royden

- Kobayashi hyperbolicity (Brody): Compact N^n is 1-hyperbolic if and only if any holomorphic map $f : \mathbb{C} \to N$ must be constant.

D1. Kobayashi hyperbolicity and the Schwarz Lemma of Yau-Royden

- Kobayashi hyperbolicity (Brody): Compact N^n is 1-hyperbolic if and only if any holomorphic map $f : \mathbb{C} \to N$ must be constant.
- Schwarz Lemma (Yau-Royden): Let $f : \Sigma \to N^n$ be a holomorphic map. Assume that the holomorphic sectional curvature of N, $H(Y) \leq -\kappa|Y|^4$ and the curvature of Riemann surface Σ, $Ric(X, \overline{X}) \geq -K|X|^2$ with $\kappa, K > 0$. Then
$$\|\partial f\|^2 \leq \frac{K}{\kappa}.$$

D1. Kobayashi hyperbolicity and the Schwarz Lemma of Yau-Royden

- Kobayashi hyperbolicity (Brody): Compact N^n is 1-hyperbolic if and only if any holomorphic map $f : \mathbb{C} \to N$ must be constant.
- Schwarz Lemma (Yau-Royden): Let $f : \Sigma \to N^n$ be a holomorphic map. Assume that the holomorphic sectional curvature of N, $H(Y) \leq -\kappa |Y|^4$ and the curvature of Riemann surface Σ, $Ric(X, \overline{X}) \geq -K|X|^2$ with $\kappa, K > 0$. Then
$$\|\partial f\|^2 \leq \frac{K}{\kappa}.$$
- It generalizes the Ahlfors's result and implies the 1-hyperbolicity of N, if N is compact and $H^N(Y) < 0$. Hence $H < 0$ is a very restrictive condition. Conditions $S_k < 0$, and $Ric_k < 0$ for $k \geq 2$ are more flexible.

E1. A new Schwarz Lemma

E1. A new Schwarz Lemma

- (Royden): *Let* $f : M^m \to N^n$ *be a holomorphic map. Assume that* $H^N(Y) \leq -\kappa |X|^4$ *and* $Ric^M(X, \overline{X}) \geq -K|X|^2$ *with* $\kappa, K > 0$. *Then with* $d = rank(f)$

$$\|\partial f\|^2 \leq \frac{2d}{d+1}\frac{K}{\kappa}.$$

E1. A new Schwarz Lemma

- (Royden): *Let $f : M^m \to N^n$ be a holomorphic map. Assume that $H^N(Y) \leq -\kappa |X|^4$ and $Ric^M(X, \overline{X}) \geq -K|X|^2$ with $\kappa, K > 0$. Then with $d = rank(f)$*
$$\|\partial f\|^2 \leq \frac{2d}{d+1}\frac{K}{\kappa}.$$
- (Yau): $\|\partial f\|^2 \leq \frac{K}{\kappa}$ was proved for the case $m \geq 2$ if bisectional curvature of N is bounded from above by $-\kappa$.

E1. A new Schwarz Lemma

- (Royden): *Let $f: M^m \to N^n$ be a holomorphic map. Assume that $H^N(Y) \leq -\kappa|X|^4$ and $Ric^M(X,\overline{X}) \geq -K|X|^2$ with $\kappa, K > 0$. Then with $d = rank(f)$*
$$\|\partial f\|^2 \leq \frac{2d}{d+1}\frac{K}{\kappa}.$$
- (Yau): $\|\partial f\|^2 \leq \frac{K}{\kappa}$ was proved for the case $m \geq 2$ if bisectional curvature of N is bounded from above by $-\kappa$.

- Theorem (N-2018)

Let (M,g) be a complete Kähler manifold such that the holomorphic sectional curvature $H^M(X)/|X|^4 \geq -K$, and (N^n,h) be a Kähler manifold with $H^N(Y) < -\kappa|Y|^4$ for some $\kappa > 0$. Let $f: M \to N$ be a holomorphic map. Then

$$\|\partial f\|_m^2 \leq \frac{K}{\kappa}, \tag{0.1}$$

provided that the bisectional curvature of M is bounded from

E2. Notation and proof

E2. Notation and proof

E2. Notation and proof

- For the tangent map $\partial f : T'_x M \to T'_{f(x)} N$ we define its maximum norm square to be

$$\|\partial f\|_m^2 \doteqdot \sup_{v \neq 0} \frac{|\partial f(v)|^2}{|v|^2}.$$

E2. Notation and proof

- For the tangent map $\partial f : T'_x M \to T'_{f(x)} N$ we define its maximum norm square to be

$$\|\partial f\|_m^2 \doteqdot \sup_{v \neq 0} \frac{|\partial f(v)|^2}{|v|^2}.$$

- It is a reminiscent of the well-known Pogorelov's Lemma for Monge-Ampère equation, since the maximum eigenvalue of $\nabla^2 u$ is the $\|\cdot\|_m$ for the differential of normal map ∇u for any smooth u.

E2. Notation and proof

- For the tangent map $\partial f : T'_x M \to T'_{f(x)} N$ we define its maximum norm square to be

$$\|\partial f\|_m^2 \doteqdot \sup_{v \neq 0} \frac{|\partial f(v)|^2}{|v|^2}.$$

- It is a reminiscent of the well-known Pogorelov's Lemma for Monge-Ampère equation, since the maximum eigenvalue of $\nabla^2 u$ is the $\| \cdot \|_m$ for the differential of normal map ∇u for any smooth u.
- It is perhaps the most natural high dimensional generalization of Ahlfors' (and Yau-Royden) Schwarz lemma for mapping between Riemann surfaces.

E2. Notation and proof

- For the tangent map $\partial f : T'_x M \to T'_{f(x)} N$ we define its maximum norm square to be

$$\|\partial f\|_m^2 \doteqdot \sup_{v \neq 0} \frac{|\partial f(v)|^2}{|v|^2}.$$

- It is a reminiscent of the well-known Pogorelov's Lemma for Monge-Ampère equation, since the maximum eigenvalue of $\nabla^2 u$ is the $\| \cdot \|_m$ for the differential of normal map ∇u for any smooth u.
- It is perhaps the most natural high dimensional generalization of Ahlfors' (and Yau-Royden) Schwarz lemma for mapping between Riemann surfaces.
- The proof uses a viscosity consideration from PDE theory. The key is to construct a smooth barrier.

E3. A geometric implication

E3. A geometric implication

E3. A geometric implication

- A classical result asserts: *A simply-connected Kähler manifold M^m with negative holomorphic sectional curvature* -1 *must be isometric to the complex hyperbolic space form* $\mathbb{CH}^m$.

E3. A geometric implication

- A classical result asserts: *A simply-connected Kähler manifold M^m with negative holomorphic sectional curvature -1 must be isometric to the complex hyperbolic space form $\mathbb{CH}^m$.*
- Naturally one could ask if the holomorphic sectional curvature is close to -1, can one prove that the metric is close to the complex hyperbolic one? Namely the 'stability' of H? (Rauch showed the topological stability for the sectional curvature, which led to quarter-pinching theorems).

E3. A geometric implication

- A classical result asserts: *A simply-connected Kähler manifold* M^m *with negative holomorphic sectional curvature* -1 *must be isometric to the complex hyperbolic space form* $\mathbb{CH}^m$.
- Naturally one could ask if the holomorphic sectional curvature is close to -1, can one prove that the metric is close to the complex hyperbolic one? Namely the 'stability' of H? (Rauch showed the topological stability for the sectional curvature, which led to quarter-pinching theorems).
- Corollary: *the equivalence of the negativities of the holomorphic sectional curvature implies the equivalence of the metrics*.

E3. A geometric implication

- A classical result asserts: *A simply-connected Kähler manifold M^m with negative holomorphic sectional curvature -1 must be isometric to the complex hyperbolic space form $\mathbb{CH}^m$.*
- Naturally one could ask if the holomorphic sectional curvature is close to -1, can one prove that the metric is close to the complex hyperbolic one? Namely the 'stability' of H? (Rauch showed the topological stability for the sectional curvature, which led to quarter-pinching theorems).
- Corollary: *the equivalence of the negativities of the holomorphic sectional curvature implies the equivalence of the metrics.*
- If two Kähler metrics g_1 and g_2 satisfy that

$$-L_1|X|^4_{g_1} \le H_{g_1}(X) \le -U_1|X|^4_{g_1}, \quad -L_2|X|^4_{g_2} \le H_{g_2}(X) \le -U_2|X|^4_{g_2}$$

then for any $v \in T'_x M$ we have the estimates:

$$|v|^2_{g_2} \le \frac{L_1}{U_2}|v|^2_{g_1}; \quad |v|^2_{g_1} \le \frac{L_2}{U_1}|v|^2_{g_2}.$$

F1. A question on k-hyperbolicity

F1. A question on k-hyperbolicity

- k-hyperbolicity: N is k-hyperbolic if and only if any holomorphic $f : \mathbb{C}^k \to N$ must be degenerate (namely $dim(f(M)) < k$).

F1. A question on k-hyperbolicity

- k-hyperbolicity: N is k-hyperbolic if and only if any holomorphic $f : \mathbb{C}^k \to N$ must be degenerate (namely $dim(f(M)) < k$).
- **Conjecture**: *Let N^n $(n \geq 2)$ be a compact Kähler manifold with $S_k < 0$. Then N^n is k-hyperbolic.*

F1. A question on k-hyperbolicity

- k-hyperbolicity: N is k-hyperbolic if and only if any holomorphic $f : \mathbb{C}^k \to N$ must be degenerate (namely $dim(f(M)) < k$).
- **Conjecture**: *Let N^n $(n \geq 2)$ be a compact Kähler manifold with $S_k < 0$. Then N^n is k-hyperbolic.*

- Theorem (N-2018)

(i) Assume that $\dim_{\mathbb{C}} M = m \leq n = \dim_{\mathbb{C}} N$. Let (M, g) be a compact Kähler manifold such that $Ric^M \geq 0$. Let (N^n, h) be a complete Kähler manifold such that $S^N_m(y) < 0$. Then any holomorphic map $f : M \to N$ must be degenerate. If the condition is relaxed to allow $S^N_m \leq 0$ then either the map is degenerate or it must be totally geodesic.
(ii) If $Ric^N_k < 0$ and N is compact, then N is k-hyperbolic. Same rigidity as (i) holds if $<$ is $\leq$.

F1. A question on k-hyperbolicity

- k-hyperbolicity: N is k-hyperbolic if and only if any holomorphic $f : \mathbb{C}^k \to N$ must be degenerate (namely $dim(f(M)) < k$).
- **Conjecture**: *Let N^n $(n \geq 2)$ be a compact Kähler manifold with $S_k < 0$. Then N^n is k-hyperbolic.*

- Theorem (N-2018)

  (i) Assume that $\dim_{\mathbb{C}} M = m \leq n = \dim_{\mathbb{C}} N$. Let (M, g) be a compact Kähler manifold such that $Ric^M \geq 0$. Let (N^n, h) be a complete Kähler manifold such that $S_m^N(y) < 0$. Then any holomorphic map $f : M \to N$ must be degenerate. If the condition is relaxed to allow $S_m^N \leq 0$ then either the map is degenerate or it must be totally geodesic.
  (ii) If $Ric_k^N < 0$ and N is compact, then N is k-hyperbolic. Same rigidity as (i) holds if $<$ is $\leq$.

  - For (ii) the equal dimensional case was known (Mok-Yau). Here $Ric_k^N < 0$ is a stronger assumption than $S_k < 0$.

F2. Related questions for Kähler manifolds with $S_k < 0$ and $Ric_k \leq 0$ ($Ric_k \geq 0$)

F2. Related questions for Kähler manifolds with $S_k < 0$ and $Ric_k \leq 0$ ($Ric_k \geq 0$)

- What can be said for $\pi_1(M)$ if $Ric_k \geq 0$? When $k = n$, Cheeger-Gromoll applies. It can not contain a surface group for $g(\Sigma) \geq 2$ (Ni-2019).

F2. Related questions for Kähler manifolds with $S_k < 0$ and $Ric_k \leq 0$ ($Ric_k \geq 0$)

- What can be said for $\pi_1(M)$ if $Ric_k \geq 0$? When $k = n$, Cheeger-Gromoll applies. It can not contain a surface group for $g(\Sigma) \geq 2$ (Ni-2019).
- Wu-Yau, Tosatti-Yang: A compact Kähler manifold with $H < 0$ (or even quasi-negative) is projective and K_M is ample.

F2. Related questions for Kähler manifolds with $S_k < 0$ and $Ric_k \leq 0$ ($Ric_k \geq 0$)

- What can be said for $\pi_1(M)$ if $Ric_k \geq 0$? When $k = n$, Cheeger-Gromoll applies. It can not contain a surface group for $g(\Sigma) \geq 2$ (Ni-2019).
- Wu-Yau, Tosatti-Yang: A compact Kähler manifold with $H < 0$ (or even quasi-negative) is projective and K_M is ample.
- Questions: how about compact Kähler manifolds with $S_2 < 0$? Are they projective? When can one deduce that K_M is ample?

F2. Related questions for Kähler manifolds with $S_k < 0$ and $Ric_k \leq 0$ ($Ric_k \geq 0$)

- What can be said for $\pi_1(M)$ if $Ric_k \geq 0$? When $k = n$, Cheeger-Gromoll applies. It can not contain a surface group for $g(\Sigma) \geq 2$ (Ni-2019).
- Wu-Yau, Tosatti-Yang: A compact Kähler manifold with $H < 0$ (or even quasi-negative) is projective and K_M is ample.
- Questions: how about compact Kähler manifolds with $S_2 < 0$? Are they projective? When can one deduce that K_M is ample?
- Similar question can be asked for $Ric_k < 0$ for $k \in (1, n)$, which is stronger than $S_k < 0$.

F2. Related questions for Kähler manifolds with $S_k < 0$ and $Ric_k \leq 0$ ($Ric_k \geq 0$)

- What can be said for $\pi_1(M)$ if $Ric_k \geq 0$? When $k = n$, Cheeger-Gromoll applies. It can not contain a surface group for $g(\Sigma) \geq 2$ (Ni-2019).
- Wu-Yau, Tosatti-Yang: A compact Kähler manifold with $H < 0$ (or even quasi-negative) is projective and K_M is ample.
- Questions: how about compact Kähler manifolds with $S_2 < 0$? Are they projective? When can one deduce that K_M is ample?
- Similar question can be asked for $Ric_k < 0$ for $k \in (1, n)$, which is stronger than $S_k < 0$.
- (Motivated by my metric stability result) Is a compact Kähler manifold with H close to -1 biholomorphic to a quotient of complex hyperbolic space? (Negative for Riemannian case by Gromov-Thurston. But true for the positive case due to Mori, Siu-Yau's result and the curvature pinching).

F3. Curvature characterization of homogenous Kähler manifolds

F3. Curvature characterization of homogenous Kähler manifolds

- HSSs are characterized by $B \geq 0$ by Mori, Siu-Yau, Mok of compact HSSs. HSSs are special homogenous Kähler spaces (C-spaces).

F3. Curvature characterization of homogenous Kähler manifolds

- HSSs are characterized by $B \geq 0$ by Mori, Siu-Yau, Mok of compact HSSs. HSSs are special homogenous Kähler spaces (C-spaces).
- Conjecture (Campana-Peternell, Zheng, 1993) For M Fano, nef of $T'M \to M$ is rational homogenous Kähler;
  Wu-Yau-Zheng (2009) attempted with a curvature notion QB (not working due to Chau-Tam, 2012).

F3. Curvature characterization of homogenous Kähler manifolds

- HSSs are characterized by $B \geq 0$ by Mori, Siu-Yau, Mok of compact HSSs. HSSs are special homogenous Kähler spaces (C-spaces).
- Conjecture (Campana-Peternell, Zheng, 1993) For M Fano, nef of $T'M \to M$ is rational homogenous Kähler; Wu-Yau-Zheng (2009) attempted with a curvature notion QB (not working due to Chau-Tam, 2012).
- Quadratic orthogonal bisectional curvature (QB) is defined as $\langle R, A^2 \bar{\wedge} id - A \bar{\wedge} A \rangle$ for any Hermitian symmetric tensor A. Locally QB$> 0 = \sum_{i,j} R_{i\bar{i}j\bar{j}}(a_i - a_j)^2 > 0$, for any unitary frame $\{e_i\}$, $\vec{a} \neq c\vec{1}$.

F3. Curvature characterization of homogenous Kähler manifolds

- HSSs are characterized by $B \geq 0$ by Mori, Siu-Yau, Mok of compact HSSs. HSSs are special homogenous Kähler spaces (C-spaces).
- Conjecture (Campana-Peternell, Zheng, 1993) For M Fano, nef of $T'M \to M$ is rational homogenous Kähler; Wu-Yau-Zheng (2009) attempted with a curvature notion QB (not working due to Chau-Tam, 2012).
- Quadratic orthogonal bisectional curvature (QB) is defined as $\langle R, A^2 \bar{\wedge} id - A \bar{\wedge} A \rangle$ for any Hermitian symmetric tensor A. Locally QB$> 0 = \sum_{i,j} R_{i\bar{i}j\bar{j}}(a_i - a_j)^2 > 0$, for any unitary frame $\{e_i\}$, $\vec{a} \neq c\vec{1}$.
- A step back: $Ric^{\perp}(X, \bar{X}) \doteqdot Ric(X, \bar{X}) - H(X)/|X|^2$. $(QB) > 0$ implies $Ric^{\perp} > 0$. Wang-Zheng-N: Classical C-spaces with $b_2 = 1$ satisfy $Ric^{\perp} > 0$ (unlike $QB > 0$); A Frankel type result holds; A complete classification for $\dim_{\mathbb{C}}(M) = 3$.

F3. Cross quadratic curvature

F3. Cross quadratic curvature

- For $n = 4$, Wang-Zheng-N formulated a conjecture for $Ric^{\perp} > 0$ Kähler manifolds.

F3. Cross quadratic curvature

- For $n = 4$, Wang-Zheng-N formulated a conjecture for $Ric^{\perp} > 0$ Kähler manifolds.
- $n \geq 5$, $Ric^{\perp} > 0$ perhaps is too weak to prove that it must be a C-space. It remains open even if such a manifold is Fano.

F3. Cross quadratic curvature

- For $n = 4$, Wang-Zheng-N formulated a conjecture for $Ric^{\perp} > 0$ Kähler manifolds.
- $n \geq 5$, $Ric^{\perp} > 0$ perhaps is too weak to prove that it must be a C-space. It remains open even if such a manifold is Fano.
- A step further (motivated by Calabi-Vesentini's work): On a Kähler manifold (M^n, g), let $T'M$ and $T''M$ be the holomorphic and anti-holomorphic tangent bundle of M, then CQB is a Hermitian quadratic form on linear maps $A : T''M \to T'M$:

$$\mathrm{CQB}(A) = \sum_{\alpha,\beta=1}^{n} R(A(\overline{E}_\alpha), \overline{A(\overline{E}_\alpha)}, E_\beta, \overline{E}_\beta) - R(E_\alpha, \overline{E}_\beta, A(\overline{E}_\alpha), \overline{A(\overline{E}_\beta)}$$

where R is the curvature tensor of M and $\{E_\alpha\}$ is a unitary frame of $T'M$. ^{d}CQB can be defined as well.

F3. Cross quadratic curvature

- For $n = 4$, Wang-Zheng-N formulated a conjecture for $Ric^{\perp} > 0$ Kähler manifolds.
- $n \geq 5$, $Ric^{\perp} > 0$ perhaps is too weak to prove that it must be a C-space. It remains open even if such a manifold is Fano.
- A step further (motivated by Calabi-Vesentini's work): On a Kähler manifold (M^n, g), let $T'M$ and $T''M$ be the holomorphic and anti-holomorphic tangent bundle of M, then CQB is a Hermitian quadratic form on linear maps $A : T''M \to T'M$:

$$\mathrm{CQB}(A) = \sum_{\alpha,\beta=1}^{n} R(A(\overline{E}_\alpha), \overline{A(\overline{E}_\alpha)}, E_\beta, \overline{E}_\beta) - R(E_\alpha, \overline{E}_\beta, A(\overline{E}_\alpha), \overline{A(\overline{E}_\beta)}$$

  where R is the curvature tensor of M and $\{E_\alpha\}$ is a unitary frame of $T'M$. ${}^d\mathrm{CQB}$ can be defined as well.
- $\mathrm{CQB}_1 > 0$ implies $Ric > 0$ and $Ric^{\perp} > 0$. The local rigidity of manifolds with ${}^d\mathrm{CQB} > 0$, and that all classical C-spaces with $b_2 = 1$ satisfies $\mathrm{CQB} > 0$, ${}^d\mathrm{CQB} > 0$ (Ni-2019).

F4. Two conjectures and partial progresses

F4. Two conjectures and partial progresses

- Kähler C-spaces with CQB≥ 0, ^{d}CQB> 0 and $b_2 \to \infty$ exist.

F4. Two conjectures and partial progresses

- Kähler C-spaces with CQB≥ 0, ^{d}CQB> 0 and $b_2 \to \infty$ exist.

- Conjecture

  Any Kähler C-space will have nonnegative CQB and positive ^{d}CQB.

F4. Two conjectures and partial progresses

- Kähler C-spaces with CQB≥ 0, ^{d}CQB> 0 and $b_2 \to \infty$ exist.

- Conjecture

  Any Kähler C-space will have nonnegative CQB and positive ^{d}CQB.

- Conjecture

  Let (M, g) be a Kähler-Einstein manifold with CQB ≥ 0 and (or) ^{d}CQB > 0. Then M is biholomorphic to a Kähler C-space.

F4. Two conjectures and partial progresses

  - Kähler C-spaces with CQB≥ 0, ^{d}CQB> 0 and $b_2 \to \infty$ exist.
- Conjecture

  Any Kähler C-space will have nonnegative CQB and positive ^{d}CQB.
- Conjecture

  Let (M, g) be a Kähler-Einstein manifold with CQB ≥ 0 and (or) ^{d}CQB > 0. Then M is biholomorphic to a Kähler C-space.
  - The conjecture without the KE condition also makes sense.

F4. Two conjectures and partial progresses

- Kähler C-spaces with CQB≥ 0, ^{d}CQB> 0 and $b_2 \to \infty$ exist.

- Conjecture

  Any Kähler C-space will have nonnegative CQB and positive ^{d}CQB.

- Conjecture

  Let (M, g) be a Kähler-Einstein manifold with CQB ≥ 0 and (or) ^{d}CQB > 0. Then M is biholomorphic to a Kähler C-space.

  - The conjecture without the KE condition also makes sense.

- Theorem (N-Zheng, 2019)

  Let (M, g) be a compact Kähler manifold with $CQB_1 \geq 0$ (or ${}^d CQB_1 \geq 0$) and its universal cover does not contain a flat de Rham factor. Then M is Fano. In fact, the Kähler-Ricci flow will evolve the metric g to ones with positive Ricci curvature.

F4. Two conjectures and partial progresses

  - Kähler C-spaces with CQB≥ 0, ^{d}CQB> 0 and $b_2 \to \infty$ exist.
- Conjecture

  Any Kähler C-space will have nonnegative CQB and positive ^{d}CQB.
- Conjecture

  Let (M, g) be a Kähler-Einstein manifold with CQB ≥ 0 and (or) ^{d}CQB > 0. Then M is biholomorphic to a Kähler C-space.
  - The conjecture without the KE condition also makes sense.
- Theorem (N-Zheng, 2019)

  Let (M, g) be a compact Kähler manifold with $CQB_1 \geq 0$ (or ${}^d\, CQB_1 \geq 0$) and its universal cover does not contain a flat de Rham factor. Then M is Fano. In fact, the Kähler-Ricci flow will evolve the metric g to ones with positive Ricci curvature.

  There are Riemannian analogues of CQB and ^{d}CQB. The result for real cases generalizes an earlier result of Böhm-Wilking on manifolds with $K \geq 0$.