Lecture 1 – Complex Manifolds and Holomorphic Vector Bundles by L. Ni

A Complex manifold M^m is a smooth manifold such that it is covered by *holomorphic atlas* $(\varphi_i : U_i \to \mathbb{C}^n)$ with local transition mappings $\varphi_i \cdot \varphi_j^{-1}$ being holomorphic. This clearly makes it possible to discuss holomorphic functions, mappings etc.

Examples: $\mathbb{C}^m$, Riemann surfaces, complex projective space $\mathbb{P}^m$, etc. Implicit function theorem also leads to the concept of submanifolds (e.g. zero set of a non-critical holomorphic function). More examples via coverings/quotiens and surgeries.

In general, a *topological manifold* can be endowed with extra structures. Hence there are analytic manifolds, algebraic manifolds, topological manifolds, PL-manifolds, etc. The fundamental problem is to classify the simplest ones in each category. One can also ask if a weaker equivalence implies a stronger equivalence, e.g. the Poincaré conjecture.

Conjecture (uniqueness): A complex manifold diffeomorphic to $\mathbb{P}^m$ *must be biholomorphic to it.* There is also the famous question: *is there a holomorphic structure on* $\mathbb{S}^6$?

The embedding problem, which unites the extrinsic and intrinsic geometry to some degree, is also universal.

Whitney(1936): any compact differential manifold of dimension n can be smoothly embedded into $2n+1$*-dimensional Euclidean space* $\mathbb{R}^{2n+1}$. This has motivated several important developments including *Nash(1956): any* C^∞ *Riemannian manifold* (M^n, g) *can be isometrically embedded into some Euclidean space* $\mathbb{R}^d$ *with* $d = d(n)$.

Unfortunately it fails completely for compact complex manifolds due to that *any holomorphic function on a compact complex manifold must be a constant.* On the other hand in this direction there are two important results:

Remmert(1956)-Narasimhan(1960)-Bishop(1961, joined UCSD afterwards): every Stein manifold M^m *can be holomorphically embedded into* $\mathbb{C}^{2m+1}$.

Kodaira (1954): every Hodge manifold can be holomorphically embedded into $\mathbb{P}^N$. This was before Nash's isometric embedding, after Nash's *real algebraic manifolds* (1951).

Two results below are related to Serre's famous GAGA principle.

Morrey-Grauert(1958): There exists an analytic embedding of the real analytic manifolds. Nash (1966): The isometric embedding is real analytic if the Riemannian manifold and the metric are real analytic.

For the concept of Hodge manifolds we need some background on the cohomology and differential forms, which we shall introduce later. First we remark that the Kodaira's embedding theorem holds a very important place in complex geometry for reasons: 1) Its proof relies on a cohomology vanishing theorem, which starts a direction/approach to study the complex/algebraic manifolds, and the method of proving this vanishing theorem relies on an estimate, from which grows the theory of L^2-estimate of $\bar{\partial}$-operator; 2) Due to a result of *Chow (1949): any complex/holomorphic submanifolds of* $\mathbb{P}^m$ *must be algebraic, namely are zeros of homogenous polynomials*, the embedding theorem connects the study of Hodge manifolds with the algebraic geometry. Hence complex geometry is naturally connected with complex analysis (by the very definition of complex manifolds), the algebra/algebraic geometry, topology, etc. The emphasis here is the geometric method aided by PDEs.

Our first goal is to prove Kodaira's theorem and more generally the vanishing theorem of Calabi-Vesentini for cohomolgy valued in a holomorphic vector bundle. We begin by getting our hands dirty with differential forms.

Differential forms of (p,q)-type (in a local coordinate chart):

$$\varphi = \frac{1}{p!q!}\sum \varphi_{A_p\bar{B}_q} dz^{\alpha_1}\wedge\cdots\wedge dz^{\alpha_p}\wedge d\bar{z}^{\beta_1}\wedge\cdots\wedge d\bar{z}^{\beta_q}.$$

with $A_p=(\alpha_1,\cdots,\alpha_p)$, $B_q=(\beta_1,\cdots,\beta_q)$, and summation is for all (p,q)-tuples.

Here $dz^\alpha = dx^\alpha+\sqrt{-1}dy^\alpha$, if $z_\alpha = x_\alpha+\sqrt{-1}y_\alpha$, $d\bar{z}^\alpha=\overline{dz^\alpha}$, $\frac{\partial}{\partial z^\alpha}=\frac{1}{2}\left(\frac{\partial}{\partial x^\alpha}-\sqrt{-1}\frac{\partial}{\partial y^\alpha}\right)$. We define $\frac{\partial}{\partial \bar{z}^\alpha}$ similarly. Now since $d=\sum \frac{\partial}{\partial x^\alpha}dx^\alpha+\frac{\partial}{\partial y^\alpha}dy^\alpha$ it is easy to see that $d=\partial+\bar\partial$. If (M,g) is a Hermitian manifold such that the Kähler form $\omega_g=\frac{\sqrt{-1}}{2}g_{\alpha\bar\beta}dz^\alpha\wedge dz^{\bar\beta}$ is d-closed (namely $d\omega=0$), we call (M,g) a Kähler manifold. It can be checked that $\frac{\omega^m}{m!}$ is the volume form.

Dolbeault's Lemma: *If φ satisfies that $\bar\partial\varphi=0$ on a disk $U\subset\mathbb{C}^m$, then there exists a ψ such that $\varphi=\bar\partial\psi$.*

This is the complex version of the Poincaré lemma: *For any φ on a disk $U\subset\mathbb{R}^n$ with $d\varphi=0$, then there exists ψ such that $d\psi=\varphi$.*

Reading: Griffiths-Harris, pp 6-18 (for a quick introduction on SCV and baby geometry); You may also consult Huybrechts, 1.1 for a slightly bigger game. These are some algebraic sides of the subject, as well as backgrounds from the functions of several complex variables. Our guiding principle here is to push all algebraic aspects into the reading. Some of them are formal and almost trivial. Others can be deep with contents.

We prove the Poincaré-Lemma first (via an induction on the dimension of the disk, following Kodaira-Morrow), which implies that any *holomorphic $(p,0)$-form φ (on a disk) is d-closed if any only if there exists a holomorphic $(p-1,0)$-form ψ such that $d\psi=\varphi$*, as well as the De Rham's theorem.

Theorem 0.1 (De Rham)**.** *The sheaf cohomology satisfies*

$$H^q(M,\mathbb{C})=\frac{H^0(M,d\mathcal{A}^{q-1})}{dH^0(M,\mathcal{A}^{q-1})}.$$

Here $\mathcal{A}^q$ is the germ of the smooth q-forms. For the proof of De Rham theorem (once equipped with the Poincaré Lemma it is almost trivial via some formality) **read** Griffiths-Harris Section 3 of Ch 0. In the token of sheaf, it follows from that the *long exact* (the exactness follows from the Poincaré Lemma) sequence:

$$0\longrightarrow\mathbb{C}\stackrel{\iota}{\longrightarrow}\mathcal{A}^0\stackrel{d}{\longrightarrow}\mathcal{A}^1\stackrel{d}{\longrightarrow}\cdots\stackrel{d}{\longrightarrow}\mathcal{A}^n\stackrel{d}{\longrightarrow}0$$

is a *fine* resolution of $\mathbb{C}$. Similarly Dolbeault's lemma (whose proof can be done in a similar fashion as Poincaré Lemma via a slightly different induction on the dimension of $d\bar{z}^\alpha$-variables, with an additional help of one variable complex analysis via the generalized Cauchy's integral formula, see page 2 of Griffiths-Harris) implies that the sheaf cohomology is the same as the cohomology induced via $\bar\partial$-operator on $\Omega^{p,q}$, the space of all smooth (p,q)-forms.

Theorem 0.2 (Dolbeault)**.** *The sheaf cohomology satisfies*

$$H^q(M,\Omega^p)=\frac{H^0(M,\bar\partial\mathcal{A}^{p,q-1})}{\bar\partial H^0(M,\mathcal{A}^{p,q-1})}.$$

Here the key is that two long *exact* sequences are the resolution of $\mathcal{O}$ (germ of holomorphic functions) and Ω^p (germ of holomorphic $(p,0)$-forms are fine:

$$0 \longrightarrow \mathcal{O} \xrightarrow{\iota} \mathcal{D} \xrightarrow{\bar\partial} \mathcal{A}^{0,1} \cdots \xrightarrow{\bar\partial} \mathcal{A}^{0,m} \xrightarrow{\bar\partial} 0;$$

$$0 \longrightarrow \Omega^p \xrightarrow{\iota} \mathcal{A}^{p,0} \xrightarrow{\bar\partial} \mathcal{A}^{p,1} \cdots \xrightarrow{\bar\partial} \mathcal{A}^{p,m} \xrightarrow{\bar\partial} 0.$$

A holomorphic vector bundle $\pi : E \to M$ *over a complex manifold* M, is a complex vector bundle such that its transitional linear map $\psi_{ij} = \psi_i \cdot \psi_j^{-1}$ induced by the local trivializations ($\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^r$) depends on points $z \in M$ holomorphically. This endows a holomorphic structure on E and implies that π. It allows the holomorphic sections and holomorphic morphisms between the bundles.

Similarly all the linear algebraic constructions such as tensor product, direct sum, exterior power, dual, determinant , etc can all be adapted to give constructions of new vector bundles from the old ones. We also adapt a *linear algebra to bundles principle.* Simply put, any linear algebraic result holds on bundles when properly formulated. **Reading:** See Huybrechts pages 25–31 of 1.2 for a rather complete linear algebra results needed for this course.

A special case is $T'M$, the holomorphic tangent bundle. We start with the Hermitian metrics, the holomorphic compatible connection associated with it and its curvature. Before we get into that we remark that a Dolbeault type theorem holds for germs consisting of holomorphic sections of a holomorphic vector bundle E. Also note that $\bar\partial$-operator is well defined for

$$\varphi = \frac{1}{p!q!} \sum \varphi^i_{A_p, \bar B_q} dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_p} \wedge d\bar z^{\beta_1} \wedge \cdots \wedge d\bar z^{\beta_q} \otimes e_i$$

if $\{e_i\}$ is a local holomorphic basis of E. From the long exact sequence (the exactness is simply the vector version of the pervious result)

$$0 \longrightarrow \Omega^p(E) \xrightarrow{\iota} \mathcal{A}^{p,0}(E) \xrightarrow{\bar\partial} \mathcal{A}^{p,1}(E) \cdots \xrightarrow{\bar\partial} \mathcal{A}^{p,m}(E) \xrightarrow{\bar\partial} 0,$$

where $\Omega^p(E)$ denotes the germ of holomorphic $(p,0)$-forms valued in E, gives the similar result concerning the identification of the sheaf cohomology and $\bar\partial$-cohomology.

Theorem 0.3. *For complex manifold* M *and holomorphic vector bundle* E*,*

$$H^q(M, \Omega^p(E)) = \frac{H^0(M, \bar\partial \mathcal{A}^{p,q-1}(E))}{\bar\partial H^0(M, \mathcal{A}^{p,q-1}(E))}.$$

An important result of Kodaira asserts that

Theorem 0.4 (Kodaira, 1953)**.** *For a compact complex manifold,*

$$H^q(M, \Omega^p(E)) = \mathcal{H}^{p,q}(M, E).$$

Here $\mathcal{H}^{p,q}(M,E)$ *denotes the space of* $\bar\partial$*-harmonic* (p,q)*-forms valued in* E*. In particular* $h^{p,q} = H^q(M, \Omega^p(E))$*, which are called the Hodge numbers, are finite for any* $0 \le p, q \le m$*.*

To prove this result we need to introduce metrics on both M and E, and the connections associated the metrics (which are needed to define first order derivatives invariantly). The result follows from the theory of elliptic partial differential systems, which to some degree resembles the linear maps between Euclidean spaces. For this result no Kähler structure is needed. *A Hermitian metric* locally is just a positive definite Hermitian symmetric matrix $(a_{i\bar{j}})$, such that,

$$\langle e_i, \bar{e}_j\rangle = a_{i\bar{j}}, \quad \langle v, \overline{w}\rangle = v^i a_{i\bar{j}} \bar{w}^j = v^i a_i^j \bar{w}^j$$

for $v = \sum_{i=1}^r v^i e_i$, $w = \sum_{i=1}^r w^j e_j$. (Last line can be viewed as a definition of the local matrix representation (a_i^j), which is Hermitian symmetric positive definite.) A complex manifold with a Hermitian metric on $T'M$ is called a Hermitian manifold. *A connection D* is a map $D : \mathcal{A}^0 \to \mathcal{A}^1$ satisfying, for $f \in C^\infty(U), s \in \mathcal{A}^0(U)$ (here we abbreviate $\mathcal{A}^0(U, E)$, the space of local sections over U as $\mathcal{A}^0(U)$)

$$D(f \cdot s) = df \cdot s + f \cdot Ds.$$

There is a natural decomposition $D = D^{'} + D^{''}$ into $\mathcal{A}^{1,0}(E)$ and $\mathcal{A}^{0,1}(E)$. These all make sense for a complex vector bundle E. The connection D is called holomorphically compatible (abbreviated as h-compatible) if $D^{''} = \bar{\partial}$. We call D *Hermitian* if $d\langle v, \bar{w}\rangle = \langle Dv, \bar{w}\rangle + \langle v, \overline{Dw}\rangle$ for $v, w \in \mathcal{A}^0(U)$.

Proposition 0.1. *Let (E, a) be a Hermitian vector bundle. Then there exists a unique Hermitian h-compatible connection. A Hermitian manifold is Kähler if and only if this connection is also Levi-Civita.*

An immediate corollary of Kodaira's result above is a duality theorem of Serre.

Corollary 0.1 (Serre)**.** *If E^* is the dual bundle of E,*

$$H^q(M^m, \Omega^p(E)) = H^{m-q}(M^m, \Omega^{m-p}(E^*)).$$

To put this result in the context we need to discuss the dual bundle E^*. First we work out the transition mappings for E^* in terms of E. Let $\psi_U : \pi^{-1}(U) \to U \times \mathbb{C}^r$, $g_{UV} = \psi_U \cdot \psi_V^{-1}$ be the localization and transitional map for $\pi : E \to M$. Recall for a linear map (isomorphism) $A : V \to W$ between to linear spaces, $A^* : W^* \to V^*$ is defined as $A^*(w^*)(v) = w^*(A(v))$. If we pick a basis $\{v_i\}$ and $\{w_\alpha\}$ for V and W, this shows that $A^* = A^{tr}$. Hence $\psi_x : \pi^{-1}(x) \to \mathbb{C}^r$ induces $\psi_x^{E^*} : (\pi^{-1}(x))^* \to (\mathbb{C}^r)^*$ as $((\psi_x)^*)^{-1}$ (for simplicity, expressed in terms of the basis we shall write as ${}^{tr}\psi_x^{-1}$). This defines $\psi_U^{E^*} : \pi^-(U) = E^*|_U \to U \times (\mathbb{C}^r)^*$. Simple calculation then shows that

$$g_{UV}^{E^*} \doteqdot \psi_U^{E^*} \cdot (\psi_V^{E^*})^{-1} =^{tr} g_{UV}^{-1}.$$

For $\varphi \in \mathcal{A}^{p,q}(E), \psi \in \mathcal{A}^{p',q'}(E^*)$ define the paring

$${}^{tr}\varphi \wedge \psi = \omega^i \wedge \sigma_i, \qquad \text{if } \varphi = \omega^i \otimes e_i; \psi = \sigma_i \otimes e^{*i}.$$

If $e_i^V = (\psi_V)^{-1}(E_i)$ and $s = a^i e_i^V$ with $E_i = (0, \cdots . \overset{i}{1}, \cdots, 0)^{tr}$. Then for $e_i^U = (\psi_U)^{-1}(E_i)$ and $s = b^i e_i^U$, we have $\psi_V(s) = \vec{a}$ and $\psi_U(s) = \vec{b}$. Hence $\vec{b} = g_{UV}(\vec{a})$. Hence if we change basis we would have $\varphi = \tilde{\omega}^i e_i'$, $(\tilde{\omega}^1, \cdots, \tilde{\omega}^r) = (\omega^1, \cdots, \omega^r) \cdot (g_{UV})^{tr}$. But if we write the corresponding vector $\vec{\sigma} = (\sigma_1, \cdots, \sigma_r)^{tr}$ as the representation of ψ under trivialization $\psi_V^{E^*}$, by the previous discussion, $\vec{\tilde{\sigma}} =^{tr} (g_{UV})^{-1}(\vec{\sigma})$. This shows that the pairing is independent of the choice of the trivializations (namely the choice of the local frames).

This pairing together with the integration of a top (m, m)-form on the manifold provides the essential duality involved in the theorem of Serre since for $\varphi \in \mathcal{A}^{p,q}(E)$ and $\psi \in \mathcal{A}^{m-p,m-q}(E^*)$, ${}^{tr}\varphi \wedge \psi$ is a (m, m)-form.

Here we prove Corollary 1 of L1. Recall from the last lecture that $\langle v, \bar{w}\rangle = \overline{\vec{w}}^{tr} \cdot a(\vec{v})$ or $\overline{\vec{w}}^{tr} \cdot a \cdot \vec{v}$. Tracing the computation of the last lecture we derive that

$$a^V =^{tr} \overline{g_{UV}} \cdot a^U \cdot g_{UV}. \tag{1}$$

Conversely one can view a Hermitian metric as positive definite Hermitian symmetric matrices a^U satisfying the above equation. This implies that $(a^*)^U = \overline{(a^U)^{-1}} =^{tr} (a^U)^{-1}$ is the natural associated metric on the dual bundle E^*.

For the sake of duality we need to introduce the operator # which is a map from $\mathcal{A}^{p,q}(E)$ to $\mathcal{A}^{q,p}(E^*)$, defined locally for $\varphi = \sum \varphi^i e_i$ as

$$\#(\varphi) = \sum_j \overline{a_i^j \varphi^i} e^{*j} = \sum_j a_{j\bar{i}} \overline{\varphi^i} e^{*j}.$$

In terms of the column vectors the map is sending φ^U to $\overline{a^U \cdot \varphi^U}$. The equation (1) can be applied to verify that this is well-defined. Indeed under two trivializations the definition yields $\psi^U = \overline{a^U \varphi^U}$ and $\psi^V = \overline{a^V \varphi^V}$. Then using (1), the fact that $g^{E^*}_{UV} =^{tr} g_{UV}$ and $\psi^V = \overline{a^V \varphi^V} =^{tr} g_{UV} \cdot \overline{a^U \cdot g_{UV}} \cdot \overline{\varphi^V}$, we have

$$^{tr}{g_{UV}}^{-1} \cdot \psi^V = \overline{a^U \cdot g_{UV}} \cdot \overline{\varphi^V} = \overline{a^U} \cdot \overline{\varphi^U} = \psi^U.$$

It is also clear that $\# \cdot \#$ is the identity map since $\overline{(a^*)^U \cdot \psi^U} = (a^U)^{-1} \cdot a^U \cdot \varphi^U = \varphi^U$.

We define $\partial_E : \mathcal{A}^{p,q}(E) \to \mathcal{A}^{p+1,q}(E)$ (which will be abbreviated as ∂) as follows:

$$\partial_E \varphi = \#(\bar{\partial}(\#(\varphi))), \quad \text{when context is clear denoted as } \partial\varphi.$$

This definition clearly coincides with the regular ∂ when E is trivial.

The connection is needed to understand ∂ invariantly. First for $v = \sum_{i=1}^r v^i e_i$, we let $De_i = \theta_i^j e_j$ (which we write as $De = e \cdot \theta$, viewing $e = (e_1, \cdots, e_r)$ as a row vector), thus have $Dv = dv^i e_i + v^i \theta_i^j e_j = e_i(dv^i + \theta_j^i v^j)$. In terms of $\vec{v} = (v^1, \cdots, v^r)^{tr}$ (the column vector) this can be expressed as $D\vec{v} = d\vec{v} + \theta \cdot \vec{v}$.

We insert the proof of Proposition 1 in L1. Direct calculation shows that

$$\partial(a_i^j) + \bar{\partial}(a_i^j) = da_i^j = d\langle e_i, \bar{e}_j\rangle = \langle \theta_i^k e_k, \bar{e}_j\rangle + \langle e_i, \overline{\theta_j^l e_l}\rangle = \theta_i^k a_k^j + a_i^l \overline{\theta_j^l}.$$

Namely $da = a \cdot \theta + \bar{\theta}^{tr} \cdot a$. If we choose a local holomorphic frame $\{e_i\}$, since D'' is the same as $\bar{\partial}$ and $\bar{\partial} e_i = 0$, $De_i = \theta_i^j e_j$ with θ_i^j being a matrix of $(1,0)$-forms. This implies that $\theta_i^k = (a^{-1})_j^k (\partial a_i^j)$. Hence it implies the uniqueness of the Hermitian h-compatible connection.

However, if the frame is not holomorphic the matrix-valued form θ may not be of $(1,0)$ type. In particular if we choose $\{e_i\}$ to be unitary, it can be checked that

$$\bar{\theta}^{tr} = -\theta.$$

If we write $v = v^i e_i$ with respect to a trivialization ψ_U as the last lecture, the convention we adapt here is viewing $v^U = (v^1, \cdots, v^r)^{tr}$ as a column vector and the summation can be viewed as $(e_1, e_2, \cdots, e_r)\cdot(v^1, \cdots, v^r)^{tr}$. The connection extends to $\mathcal{A}^d \to \mathcal{A}^{d+1}$, defined as for $\sigma \in \mathcal{A}^0(E)$ and $\varphi \in \mathcal{A}^d$,

$$D(\sigma\cdot\varphi) = D\sigma\wedge\varphi + \sigma\cdot d\varphi.$$

For a holomorphic σ we have $D(\sigma\cdot\varphi) = D'\sigma\wedge\varphi + \sigma\cdot d\varphi$ and $D'(\sigma\cdot\varphi) = D'\sigma\wedge\varphi + \sigma\cdot\partial\varphi$, which send $\mathcal{A}^{p,q}(E)\to\mathcal{A}^{p+1,q}(E)$. And $D''(\sigma\cdot\varphi) = \sigma\cdot\bar\partial\varphi$ coincide with $\bar\partial$.

We now can write ∂ in terms of the connection:

$$(\partial\varphi)^U = \partial\varphi^U + \theta\wedge\varphi^U, \text{ or invariantly } \partial_E = D'. \tag{2}$$

Besides ∂-operator, for the Serre's duality we also need another basic operator, namely the Hodge $*$ operator (also defined for any Riemannian manifold), which maps $\mathcal{A}^{p,q}(M)$ (namely (p,q)-forms) into $\mathcal{A}^{m-q,m-p}(M)$ (namely $(m-q,m-p)$-forms). **Reading:** Huybrechts, Section 1.2 for a rather comprehensive coverage of this operator. This operator naturally extends to an isomorphism of $\mathcal{A}^{p,q}(E)\to\mathcal{A}^{m-q,m-p}(E)$. This operator only involves the metric on the manifold M, has nothing to do with any additional structure on E.

Proposition 0.1. *The $*$-operator defined with respect to $\frac{\omega^m}{m!}$ satisfies that*

$$\langle\varphi,\bar\psi\rangle\frac{\omega^m}{m!} = \varphi\wedge *\bar\psi; \quad \overline{*\psi} = *\bar\psi; \quad **\psi = (-1)^{p+q}\psi, \text{ for } \psi\in\mathcal{A}^{p,q}.$$

Moreover $*$ is a linear isomorphism between $\mathcal{A}^{p,q}$ and $\mathcal{A}^{m-q,m-p}$.

The $*$ operator extends to $\mathcal{A}^{p,q}(E)$ as $*\varphi = \sum_{i=1}^r *\varphi^i\cdot e_i$ if $\varphi = \sum_{i=1}^r \varphi^i\cdot e_i$.

Using the pairing, $\#$ and $*$ operators (clearly $\#\cdot * = *\cdot\#$) we define a Hermitian inner product for $\varphi,\psi\in\mathcal{A}^{p,q}(E)$, (write as $(\varphi,\psi) = \langle\varphi,\bar\psi\rangle$) by the equation:

$$(\varphi,\psi)\frac{\omega^m}{m!} \doteqdot {}^{tr}\varphi\wedge *(\#(\psi)).$$

We prove below that $*$ is an isometry, $\#$ is a conjugate isometry.

Proposition 0.2.

$$(\varphi,\psi) = (*\varphi,*\psi) = (\#\psi,\#\varphi).$$

Proof. We first observe that for $\sigma,\gamma\in\mathcal{A}^{p,q}$, $\bar\sigma\wedge *\gamma = \gamma\wedge *\bar\sigma$ since $*(dx^1\wedge\cdots\wedge dx^d) = dx^{d+1}\wedge\cdots\wedge dx^{2m}$ and $*^2 = (-1)^{d(2m-d)} = (-1)^d$. Writing in a slightly different way we have $\bar\sigma\wedge *\gamma = (-1)^{p+q} *\bar\sigma\wedge\gamma$. Or $\sigma\wedge *\bar\gamma = (-1)^{p+q} *\sigma\wedge\bar\gamma$.

Now $(*\varphi,*\psi) =^{tr} *\varphi\wedge *\# *\psi = (-1)^{p+q\,tr} *\varphi\wedge\#\psi = {}^{tr}\varphi\wedge *\#\psi = (\varphi,\psi)$.

Similarly $(\#\varphi,\#\psi) =^{tr} \#\varphi\wedge\#\# *\psi = {}^{tr}\#\varphi\wedge *\psi = (-1)^{p+q} *{}^{tr}\#\varphi\wedge\psi = (\psi,\varphi)$. □

Using this we also define a L^2-Hermitian inner product (by abusing we use the same notations):

$$(\varphi,\psi) = \langle\varphi,\bar\psi\rangle = \int_M {}^{tr}\varphi\wedge *(\#(\psi)) = \int_M (\varphi,\psi)\frac{\omega^m}{m!}.$$

For the pairing (defined last lecture) of $\varphi\in\mathcal{A}^{p,q}(E)$ and $\psi\in\mathcal{A}^{p',q'}(E)$ it holds

$$\bar\partial({}^{tr}\varphi\wedge\psi) =^{tr} (\bar\partial\varphi)\wedge\psi + (-1)^{p+q}\quad {}^{tr}\varphi\wedge\bar\partial\psi.$$

In particular, if $\varphi \in \mathcal{A}^{p,q-1}(E)$ and $\psi \in \mathcal{A}^{m-p,m-q}(E^*)$, we have that

$$\int_M {}^{tr}(\bar\partial\varphi)\wedge\psi = (-1)^{p+q}\int_M {}^{tr}\varphi\wedge\bar\partial\psi.$$

Now we define $\bar\partial^* = -*\partial*$, and $\partial^* = -*\bar\partial*$. Using the above equaiton we can show that the notations are justified since they are the adjoint operators of $\bar\partial$ and ∂ respectively. Indeed

$$\begin{aligned}(\bar\partial\varphi,\psi) &= \int_M {}^{tr}(\bar\partial\varphi)\wedge *\#\psi = (-1)^{p+q}\int_M {}^{tr}\varphi\wedge\bar\partial\#*\psi \\ &= -\int_M {}^{tr}\varphi\wedge **\#\#\bar\partial\#*\psi = \int_M {}^{tr}\varphi\wedge *\#(-*\partial*\psi) \\ &= (\varphi,\bar\partial^*\psi),\ \text{for}\ \varphi\in\mathcal{A}^{p,q-1}(E),\psi\in\mathcal{A}^{p,q}(E).\end{aligned}$$

Now we define $\Box_{\bar\partial} \doteqdot \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$, and $\Box_\partial \doteqdot \partial\partial^* + \partial^*\partial$. The $\mathcal{H}^{p,q}_{\bar\partial}(E)$ is defined to be all φ, (p,q)-forms valued in E which are harmonic, namely $\Box_{\bar\partial}\varphi = 0$. Similarly we define the space $\mathcal{H}^{p,q}_{\partial}(E)$ as the kernel of the operator $\Box_\partial = \partial^*\partial + \partial\partial^*$. When M is compact

$$(3)\qquad \Box_{\bar\partial}\varphi = 0, \iff \bar\partial\varphi = 0 \text{ and } \bar\partial^*\varphi = 0; \qquad \Box_\partial\psi = 0, \iff \partial\psi = 0 \text{ and } \partial^*\psi = 0.$$

To deduce Serre's result (Corollary 1 of L1) from Kodaira's (Theorem 4 of L1) we need the following two identities:

$$(4)\qquad \partial = \#\cdot\bar\partial\cdot\#, \quad or \quad \#\cdot\partial = \bar\partial\cdot\#; \quad \#\cdot\bar\partial^* = \partial^*\cdot\#.$$

The first is the definition, the second one follows from that $\partial^*\# = -*\bar\partial*\# = -*\#\#\bar\partial\#* = -\#*\partial* = \#\bar\partial^*$. Hence $\#$ provides an isomorphism between $\mathcal{H}^{p,q}_{\bar\partial}(E)$ and $\mathcal{H}^{q,p}_{\partial}(E^*)$. Similarly using $\bar\partial^* = -*\partial*$, and $\partial^* = -*\bar\partial*$, $*$ gives an isomorphism between $\mathcal{H}^{p,q}_{\bar\partial}(E)$ and $\mathcal{H}^{m-q,m-p}_{\partial}(E)$. Putting them together we prove the Serre's duality:

$$\mathcal{H}^{p,q}_{\bar\partial}(E) \overset{\#}{\cong} \mathcal{H}^{q,p}_{\partial}(E^*) \overset{*}{\cong} \mathcal{H}^{m-p,m-q}_{\bar\partial}(E^*).$$

The curvature $R : \mathcal{A}^0(E) \to \mathcal{A}^2(E)$ is simply defined as $R \doteqdot D\cdot D$. Direct computation shows $R(v) = e_k(d\theta^k_j + \theta^k_i\wedge\theta^i_j)v^j$. Namely in terms of the column vectors $R(\vec v) = (d\theta+\theta\wedge\theta)\cdot\vec v$, which we denote as

$$\Omega = d\theta + \theta\wedge\theta.$$

On the other hand for two 1-form valued matrices a and b, it is easy to check that $(a\wedge b)^{tr} = -b^{tr}\wedge a^{tr}$. Hence, under an unitary frame we have that

$$\overline{\Omega}^{tr} = d\bar\theta^{tr} - \bar\theta^{tr}\wedge\bar\theta^{tr} = -d\theta - \theta\wedge\theta = -\Omega.$$

Given that for the holomorphic compatible case R has no $(0,2)$ component, the above implies that Ω is of $(1,1)$-type if D is a Hermitian holomorphic compatible connection. This can be seen as below. First if e' is another frame with $e = e'\cdot b$, a direct calculation shows that $\Omega = b^{-1}\Omega' b$. In particular, the type of Ω is independent of the choice of the frame. Hence with $(0,2)$ part vanishing together with $\Omega = -\overline{\Omega}^{tr}$ for a unitary frame implies the vanishing of $(2,0)$-part. The equation $\Omega = b^{-1}\Omega' b$. also implies that the symmetric functions of the eigenvalues of Ω is also independent of the choice of the frame. In terms of a local holomorphic frame we have that

$$\theta = a^{-1}\partial a, \quad \Omega = \bar{\partial}\left(a^{-1}\partial a\right). \tag{5}$$

A holomorphic *line bundle* is a holomorphic vector bundle of rank one. If we equip it with a Hermitian metric a, the curvature will be $-\partial\bar{\partial}\log a$, and we denote

$$c_1(L,a) = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log a$$

the first Chern form of (L,a). The above relation on the metrics of L and L^* shows that $c_1(L,a) = -c_1(L^*,a^*)$.

We say R is *positive in the sense of Griffiths* if for any nonzero pairs $X \in T'_xM$ and $v \in E_x$,

$$\langle R_{X\bar{X}}v, \overline{v}\rangle > 0.$$

In terms of a local trivialization with unitary frame it amounts to $\overline{v^U}^{tr} \cdot \Omega^U_{X\bar{X}} \cdot v^U > 0$

A line bundle L is called *very ample* if a basis $H^0(M,L)$, say $\{s_0, \cdots, s_N\}$ gives an embedding of M in $\mathbb{P}^N$ by

$$x \to [s_0(x), \cdots, s_N(x)] \in \mathbb{P}^N.$$

The line bundle is called *ample* if kL is very ample for some large k. The Kodaira's embedding theorem amounts to show that *any positive line bundle is ample*, since being Hodge means that there exists a Kähler metric on M such that its Kähler form is in an integral cohomology class. This implies that there exists a line bundle (L,a), $c_1(L,a) = \omega > 0$.

Lemma 0.1. *For a Hermitian holomorphic vector bundle (E,a) and its dual $(E^*, \bar{a}^{-1})$, their curvature forms satisfy that*

$$\Omega_{E^*} = -\Omega_E^{tr}.$$

Proof. Under the holomorphic basis, it can be computed that $\theta^* = (a^*)^{-1}\partial a^* = \bar{a}\partial(\overline{a^{-1}}) = -\partial\bar{a}\cdot\bar{a}^{-1}$. On the other hand $\bar{a} = a^{tr}$, we then have that

$$\theta^* = -\left(a^{-1}\partial a\right)^{tr} = -\theta^{tr}, \text{ hence } \Omega^* = \bar{\partial}\theta^* = -\bar{\partial}\theta^{tr} = -\Omega^{tr}. \tag{6}$$

Alternately, for smooth sections, v of E and w of E^* we have the pairing (v,w) which is a smooth function. Then $d(v,w) = (D_Ev, w) + (v, D_{E^*}w)$. Then

$$0 = d^2(v,w) = (D_E^2 v, w) - (D_Ev, D_{E^*}w) + (D_Ev, D_{E^*}w) + (v, D_{E^*}^2 w) = (R_Ev, w) + (v, R_{E^*}w).$$

It is easy to see that $A^* : W^* \to V^*$, defined as $(v, A^*(w^*)) = A^*(w^*)(v) = w^*(A(v)) = (A(v), w^*)$ in terms of matrix is given by A^{tr}. Hence we have the claim. □

As a corollary we see that (E,a) is G-positive if and only if $(E^*, \bar{a}^{-1})$ is G-negative. The above discussion can be applied to $E = T'M$, the holomorphic tangent bundle. This endows M with a Hermitian holomorphic compatible connection (called *Chern connection*) and a curvature (*Chern curvature*).

Using (2), for holomorphic bundle E, and $\varphi \in \mathcal{A}^{p,q}(E)$ we have ,

$$\left[\left(\partial_E\bar{\partial} + \bar{\partial}\partial_E\right)\varphi\right]^U = \Omega^U \wedge \varphi^U \doteqdot e(\Omega)(\varphi^U), \text{ invariantly } \left[D'D'' + D''D'\right]\varphi = e(R)(\varphi). \tag{7}$$

We denote by L the operator $e(2\omega_g)(\varphi)$, namely $2\omega_g \wedge (\cdot)$, with ω_g being the Kähler form.

The strategy is to apply the elliptic theory to this functional, and solve the PDE in the associated Hilbert space $H^1_{p,q}(E) = \{\varphi \in L^2_{p,q}(M,E) \,|\, \|\varphi\|^2_{H^1} < \infty\}$ with $\|\varphi\|^2_{H^1} = \|\varphi\|^2_{L^2} + \mathcal{E}(\varphi)$, and $\mathcal{E}(\varphi) = \int_M |\bar\partial\varphi|^2 + |\bar\partial^*\varphi|^2$. In the case M is compact, one in particular can view this Hilbert space as the completion of smooth ones with respect to the norm $\|\cdot\|^2_{H^1}$.

Theorem 0.1 (Kodaira)**.** *For any $\varphi \in L^2_{p,q}(M,E)$, there exists $H^1_{p,q}$, form σ, and α and β in proper spaces such that*

$$\varphi = \sigma + \bar\partial^*\alpha + \bar\partial\beta, \text{ with } \bar\partial\sigma = \bar\partial^*\sigma = 0, \bar\partial\alpha = 0 = \bar\partial^*\beta, \alpha = \bar\partial\gamma, \beta = \bar\partial^*\gamma$$

where γ solves $\Box_{\bar\partial}\gamma = \varphi - \sigma$. The decomposition is unique in the sense that if there exists another set $\{\sigma_1, \alpha_1, \beta_1\}$ then $\sigma = \sigma_1$, $\bar\partial\beta = \bar\partial\beta_1$, $\bar\partial^*\alpha = \bar\partial^*\alpha_1$.

The result has a simple version in the finite dimensional spaces. Let $D : V \to V$ be a map between Euclidean space with $D^2 = 0$. The adjoint D^* can be defined also with $(D^*)^2 = 0$. First it is easy to have, for $D : V \to V'$, the orthogonal decomposition $V' = D(V) \oplus \ker(D^*)$. Now since for $V' = V$, $D^*(V) \subset \ker(D^*)$, $D^* : V \to V$ can be viewed as a map $D^* : V \to \ker(D^*)$. Applying above with $V' = \ker(D^*)$ we have $\ker(D^*) = D(V^*) \oplus \ker(D) \cap \ker(D^*)$. One then has that $V = D(V) \oplus D^*(V) \oplus H$ with $H = \ker(D) \cap \ker(D^*)$. The above result can be better understood by considering the operator $\bar\partial$ and $\bar\partial^*$ on the the space

$$\mathcal{A}^{p,*}(E) = \mathcal{A}^{p,0}(E) \oplus \mathcal{A}^{p,1}(E) \oplus \cdots \oplus \mathcal{A}^{p,m}(E)$$

and the infinite dimensional analogue of this finite dimensional result.

To prove this decomposition result, the key step is the regularity result which asserts that for $H^1_{p,q}$-solution φ, $\Box_{\bar\partial}\varphi = \gamma$ with $\gamma \in L^2_{p,q}$, φ has improved regularity, namely $\varphi \in H^2_{p,q}$ (which means that the first derivatives of φ is in H^1). Built upon this, the regularity theory then implies that harmonic forms are smooth. The finiteness of $\dim \mathcal{H}^{p,q}$ can then be derived from this regularity (estimate) and the Rellich compactness theorem. By the theory of the elliptic PDEs, if σ is the L^2 projection of φ to $\mathcal{H}^{p,q}$, one can solve $\Box_{\bar\partial}\gamma = \varphi - \sigma$. By the regularity theory σ is always smooth. Hence if φ is smooth, γ is smooth again by the regularity theory. This is enough to imply Theorem 4 of L1. The theory also holds for suitable boundary value problems.

Once we have the decomposition result it is then easy to see that for any given class $[\eta] \in H^q(M, \Omega^p(E))$, pick a smooth $\varphi \in [\eta]$, by applying the result above, together with the regularity result which asserts that if φ is smooth then σ, α and β involved are smooth. This implies $\bar\partial^*\alpha = 0$, and $\sigma = \varphi - \bar\partial\beta$, hence is in the same cohomology class $[\eta]$. Define $\mathcal{E}'(\varphi) = \int_M (\varphi, \varphi)\frac{\omega^m}{m!}$. The energy of σ must be the minimal in the cohomology class due to the convexity of the energy $\mathcal{E}'$. The *Schauder's estimate* for the elliptic operator $\bar\partial + \bar\partial^*$ supplies more precise regularity results.

It is helpful to know how to compute locally. Let $g_{\alpha\bar\beta}$ be the Hermitian metric of the tangent bundle. The convention is $g^{\alpha\bar\beta}$ being the inverse and $\overline{g_{\alpha\bar\beta}} = g_{\beta\bar\alpha}$. In terms of the local coordinates, it can be checked that for $\varphi, \psi \in \mathcal{A}^{p,q}$,

$$(\varphi, \psi) = \frac{1}{p!q!} \sum_{A_p, B_q, \Lambda_p, N_q} g^{\bar\lambda_1\alpha_1} \cdots g^{\bar\lambda_p\alpha_p} \cdot g^{\bar\beta_1\mu_1} \cdots g^{\bar\beta_q\mu_q} \varphi_{A_p\bar B_q} \overline{\psi_{\Lambda_p \bar N_q}}.$$

If we only summing for the ordered indices then $\frac{1}{p!q!}$ is not needed. Let

$$\varphi^{\bar{A}_p B_q} = \sum_{\Lambda_p, N_q} g^{\bar{\alpha}_1 \lambda_1} \cdots g^{\bar{\alpha}_p \lambda_p} \cdot g^{\bar{\mu}_1 \beta_1} \cdots g^{\bar{\mu}_q \beta_q} \varphi_{\Lambda_p \bar{N}_q}.$$

Hence in terms of the ordered indices

$$(\varphi, \psi) = \sum_{ord} \varphi^{\bar{\Lambda}_p N_q} \overline{\psi_{\Lambda_p \bar{N}_q}} = (-1)^{pq} \sum_{ord} \varphi^{\bar{\Lambda}_p N_q} \bar{\psi}_{N_q \bar{\Lambda}_p} \tag{1}$$

noting that $\bar{\psi}_{N_q \bar{\Lambda}_p} = (-1)^{pq} \overline{\psi_{\Lambda_p \bar{N}_q}}$. For forms taken value in a holomorphic vector bundle (E, a), $\varphi = \frac{1}{p!q!} \sum \varphi^i_{A_p \bar{B}_q} dz^{A_p} \wedge d\bar{z}^{B_q} \cdot e_i$, similarly we have

$$(\varphi, \psi) = \frac{1}{p!q!} \sum_{i,j,A_p,B_q,\Lambda_p,N_q} g^{\bar{\lambda}_1 \alpha_1} \cdots g^{\bar{\lambda}_p \alpha_p} \cdot g^{\bar{\beta}_1 \mu_1} \cdots g^{\bar{\beta}_q \mu_q} \varphi^i_{A_p \bar{B}_q} \overline{\psi^j_{\Lambda_p \bar{N}_q}} a_{i\bar{j}}.$$

which can be expressed as

$$(\varphi, \psi) = \sum_{ord} \varphi^{i\bar{\Lambda}_p N_q} \overline{\psi^j_{\Lambda_p \bar{N}_q}} a^j_i = \sum_{ord} \varphi^{i\bar{\Lambda}_p N_q} \overline{\psi^j_{\Lambda_p \bar{N}_q} a^i_j} = (-1)^{pq} \sum_{ord} \varphi^{i\bar{\Lambda}_p N_q} (\#\psi)_{i N_q \bar{\Lambda}_p} \tag{2}$$

noting $(\#\psi)_{i N_q \bar{\Lambda}_p} = (-1)^{pq} \overline{\psi^j_{\Lambda_p \bar{N}_q} a^i_j} = (-1)^{pq} \overline{\psi^j_{\Lambda_p \bar{N}_q}} a_{i\bar{j}}$.

To check these one needs to verify that the inner product (defined as above) satisfies the property in Proposition 1 of last lecture. For ordered A_p let $A_{m-p} = (\alpha_{p+1}, \cdots, \alpha_m)$ with $\alpha_j < \alpha_k$ if $j < k$, and $A_p A_{m-p}$ forming a permutation of $(1, \cdots, m)$. Define $\mathrm{sgn}(\alpha)$ to be the signature of this permutation. Define B_{m-q} and $\mathrm{sgn}(\beta)$, the signature of $B_q B_{m-q}$ similarly. The $*$ operator can be obtained (in terms of the complex coordinates)

$$*\varphi = \left(\frac{\sqrt{-1}}{2}\right)^m (-1)^{\frac{m(m-1)}{2} + pm} \sum_{ord} \mathrm{sgn}(\alpha)\mathrm{sgn}(\beta) \varphi^{\bar{A}_p B_q} dz^{B_{m-q}} \wedge d\bar{z}^{A_{m-p}}.$$

Since $\bar{\psi} = \sum \bar{\psi}_{B_q \bar{A}_p} dz^{B_q} \wedge d\bar{z}^{A_p} = \sum (-1)^{pq} \overline{\psi_{A_p \bar{B}_q}} dz^{B_q} \wedge d\bar{z}^{A_p}$

$$\begin{aligned} *\bar{\psi} &= \left(\frac{\sqrt{-1}}{2}\right)^m (-1)^{\frac{m(m-1)}{2} + qm} \sum_{ord} \mathrm{sgn}(\alpha)\mathrm{sgn}(\beta) g^{\bar{\beta}_1 \mu_1} \cdots g^{\bar{\beta}_q \mu_q} g^{\alpha_1 \bar{\lambda}_1} \cdots g^{\alpha_p \bar{\lambda}_p} \\ &\quad \cdot (-1)^{pq} \overline{\psi_{\Lambda_p \bar{N}_q}} dz^{A_{m-p}} \wedge d\bar{z}^{B_{m-q}}. \end{aligned} \tag{3}$$

This implies the first equality of Proposition 1, namely that

$$\begin{aligned} \varphi \wedge *\bar{\psi} &= \left(\frac{\sqrt{-1}}{2}\right)^m (-1)^{\frac{m(m-1)}{2} + qm + pq} \sum_{ord} \varphi^{\bar{\Lambda}_p N_q} \overline{\psi_{\Lambda_p \bar{N}_q}} \\ &\quad \cdot (-1)^{q(m-p)} dz^1 \wedge dz^m \wedge d\bar{z}^1 \cdots d\bar{z}^m = (\varphi, \psi) \frac{\omega^m}{m!}. \end{aligned}$$

At the mean time, by (3), we have the second equality in Proposition 1:

$$\begin{aligned} \overline{*\psi} &= \left(\frac{\sqrt{-1}}{2}\right)^m (-1)^{\frac{m(m-1)}{2} + pm + m} \sum_{ord} \mathrm{sgn}(\alpha)\mathrm{sgn}(\beta) \overline{\psi^{\bar{A}_p B_q}} d\bar{z}^{B_{m-q}} \wedge dz^{A_{m-p}} \\ &= \left(\frac{\sqrt{-1}}{2}\right)^m (-1)^{\frac{m(m-1)}{2} + pm + m} \sum_{ord} \mathrm{sgn}(\alpha)\mathrm{sgn}(\beta) g^{\bar{\beta}_1 \mu_1} \cdots g^{\bar{\beta}_q \mu_q} g^{\alpha_1 \bar{\lambda}_1} \cdots g^{\alpha_p \bar{\lambda}_p} \\ &\quad \cdot (-1)^{(m-p)(m-q)} \overline{\psi_{\Lambda_p \bar{N}_q}} dz^{A_{m-p}} \wedge d\bar{z}^{B_{m-q}} = *\bar{\psi}. \end{aligned}$$

For computation for $\bar\partial$ and $\bar\partial^*$ in local coordinates, with $\varphi = \frac{1}{p!q!}\sum \varphi^i_{A_p\bar B_q} dz^{A_p}\wedge d\bar z^{B_q} e_i$,

$$
\begin{aligned}
\bar\partial\varphi &= \frac{1}{p!(q+1)!}\sum_{A_p,B_{q+1}} (\bar\partial\varphi)^i_{A_p\bar B_{q+1}} dz^{A_p}\wedge d\bar z^{B_{q+1}} e_i \\
&= \frac{1}{p!q!}\sum_{A_p,B_q}\sum_{\beta} \frac{\partial\varphi^i_{A_p\bar B_q}}{\partial\bar z^\beta} d\bar z^\beta \wedge dz^{A_p}\wedge d\bar z^{B_q} e_i \\
&= \frac{1}{p!(q+1)!}\sum_{A_p,B_{q+1}} (-1)^p \left(\sum_{\nu=1}^{q+1}(-1)^{\nu-1}\frac{\partial\varphi^i_{A_p\bar B^{\hat\nu}_{q+1}}}{\partial\bar z^{\beta_\nu}}\right) dz^{A_p}\wedge d\bar z^{B_{q+1}} e_i. \qquad (4)
\end{aligned}
$$

Observe that $\overline{(\varphi,\bar\partial\psi)} = (\bar\partial\psi,\varphi) = (\psi,\bar\partial^*\varphi) = \overline{(\bar\partial^*\varphi,\psi)}$ implies $(\varphi,\bar\partial\psi) = (\bar\partial^*\varphi,\psi)$. Thus

$$
\begin{aligned}
(\varphi,\bar\partial\psi) &= \frac{1}{p!(q+1)!}\sum\int_M \varphi^{i\bar\Lambda_p N_{q+1}}\overline{(\bar\partial\psi)^j_{\Lambda_p\bar N_{q+1}}}\cdot a^j_i\cdot g\,\frac{dX}{2^m} \\
&= \frac{(-1)^p}{p!(q+1)!}\sum\int_M \varphi^{i\bar\Lambda_p N_{q+1}}\sum_{\nu=1}^{q+1}(-1)^{\nu-1}\overline{\frac{\partial\psi^j_{\Lambda_p\bar N^{\hat\nu}_{q+1}}}{\partial\bar z^{\mu_\nu}}}\cdot a^j_i\cdot g\,\frac{dX}{2^m} \\
&= \frac{(-1)^{p+1}}{p!q!}\sum\int_M\sum_{\mu=1}^m \frac{\partial\left(\varphi^{i\bar\Lambda_p\mu N_q}\cdot a^j_i\cdot g\right)}{\partial z^\mu}\overline{\psi^j_{\Lambda_p\bar N_q}}\,\frac{dX}{2^m} \\
&= \frac{(-1)^{p+1}}{p!q!}\sum\int_M\sum_{\mu=1}^m\left(\frac{\partial\varphi^{j\bar\Lambda_p\mu N_q}}{\partial z^\mu} + \varphi^{i\bar\Lambda_p\mu N_q}\frac{\partial a^k_i}{\partial z^\mu}(a^{-1})^j_k + \varphi^{j\bar\Lambda_p\mu N_q}\frac{\partial g}{\partial z^\mu}\frac{1}{g}\right) \\
&\quad \cdot a^l_j\cdot g\overline{\psi^l_{\Lambda_p\bar N_q}}\,\frac{dX}{2^m}
\end{aligned}
$$

where $dX = dx^1\wedge dy^1\cdots dx^m\wedge dy^m$, $g=\det(g_{\alpha\bar\beta})$, $N^{\hat\nu}_{q+1}$ means the ν-th index being omitted. Hence

$$
(\bar\partial^*\varphi)^{j\bar\Lambda_p N_q} = (-1)^{p+1}\sum_{\mu=1}^m\left(\frac{\partial\varphi^{j\bar\Lambda_p\mu N_q}}{\partial z^\mu} + \left(\theta^j_i\right)_{\frac{\partial}{\partial z^\mu}}\varphi^{i\bar\Lambda_p\mu N_q} + \varphi^{j\bar\Lambda_p\mu N_q}\frac{\partial g}{\partial z^\mu}\frac{1}{g}\right). \qquad (5)
$$

The formulae (4) and (5) ensure that the $\Box_{\bar\partial}$ is a second order elliptic operator. At the mean time, they are also very useful in deriving some Kodaira-Bochner formulae.

The Kodaira-Hodge theorem has important applications via a criterion of Bochner, according to which a tensor of a specific type cannot satisfy a given "harmonic" equation globally unless it is identically zero, if the curvature tensor satisfies some inequality everywhere. Combining with the Kodaira-Hodge theorem, this implies the vanishing of certain cohomology groups. A simple $\partial\bar\partial$-lemma below is useful to study holomorphic sections.

Proposition 0.1. *Let (E,a) be a holomorphic vector bundle equipped with a a Hermitian metric and the compatible Hermitian connection. Then for a holomorphic section s of E*

$$\partial_X\bar\partial_{\overline{X}}\|s\|^2 = \langle D_X s,\overline{D_X s}\rangle - \langle R_{X\overline{X}}s,\bar s\rangle.$$

Proof. Direct calculation shows that, using $D'' = \bar\partial$,

$$
\begin{aligned}
\partial\bar\partial\|s\|^2 &= -\bar\partial\partial\|s\|^2 = -\bar\partial\left(\langle D's,\bar s\rangle + \langle s,\overline{\bar\partial s}\rangle\right) \\
&= -\bar\partial\langle D's,\bar s\rangle = \langle D's,\overline{D's}\rangle - \langle D''D's,\bar s\rangle.
\end{aligned}
$$

The result follows by observing that $D''D's = D^2 s = Rs$ for a holomorphic section. $\square$

The curvature R can be written as $R = R^{j}_{i\,\alpha\bar\beta} dz^\alpha \wedge d\bar z^\beta e^{*i} \otimes e_j$. We define $c_1(E,a) \doteqdot \frac{\sqrt{-1}}{2\pi}\left(\sum_{i=1}^r R^i_{i\alpha\bar\beta}\right) dz^\alpha \wedge d\bar z^\beta$ as the first Chern form of E. We also introduce the tensor $R_{i\bar k\alpha\bar\beta} \doteqdot a_{j\bar k} R^{j}_{i\,\alpha\bar\beta}$, which then defines a Hermitian form Θ on $T'M \otimes E$ as

$$\Theta(X\otimes\eta, \overline{X\otimes\eta}) \doteqdot R_{i\bar j\alpha\bar\beta} X^\alpha \overline{X^\beta} b^i \overline{b^j} = \langle R^{j}_{i\,X\overline{X}} b^i, \bar\eta\rangle \doteqdot \langle \Theta_{X\overline{X}}(\eta), \bar\eta\rangle,$$

where $X = X^\alpha \frac{\partial}{\partial z^\alpha}$ and $\eta = b^j e_j$. (Einstein convention is used for repeated indices.) The curvature is called *positive in the sense of Nakano* if for any $\tau = \tau^{i\alpha}\frac{\partial}{\partial z^\alpha}\otimes e_i \neq 0$

$$\Theta(\tau,\bar\tau) \doteqdot R_{i\bar k\alpha\bar\beta}\tau^{i\alpha}\overline{\tau^{k\beta}} > 0.$$

If M is equipped with a Hermitian metric $g_{\alpha\bar\beta}$ we define the *mean curvature*, a Hermitian form on E, for $\eta = b^j e_j$ as $\hat K(\eta,\bar\eta) \doteqdot g^{\alpha\bar\beta} R_{i\bar k\alpha\bar\beta} b^i \overline{b^k} = \langle g^{\alpha\bar\beta}\Theta_{\alpha\bar\beta}(\eta), \bar\eta\rangle$. A metric a is called *Hermitian-Einstein* if $\hat K = \lambda a$ for some $\lambda \in \mathbb{R}$, *Kähler-Einstein* if $E = T'M$ and a is Kähler. By the maximum principle and the above proposition, we have the following corollary.

Corollary 0.1. *On a compact Hermitian manifold (M,g). Assume that the holomorphic vector bundle (E,a) has quasi-negative mean curvature $\hat K$. Then $H^0(M,E) = \{0\}$.*

Recall $e(R)$ defined on $\mathcal{A}^{p,q}(E)$, which maps $\varphi = \frac{1}{p!q!}\varphi^i_{A_p\bar B_q} dz^{A_p}\wedge d\bar z^{B_q}\otimes e_i$ to

$$e(R)(\varphi) \doteqdot \frac{1}{p!q!}\sum R^{i}_{j\,\alpha\bar\beta}\varphi_{A_p\bar B_q} dz^\alpha\wedge d\bar z^\beta \wedge dz^{A_p}\wedge d\bar z^{B_q}\otimes e_i.$$

Its conjugate will be denoted as $\iota(R)$. The computation of the last lecture shows that

$$(\partial\bar\partial + \bar\partial\partial)\varphi = e(R)(\varphi). \tag{6}$$

For a Kähler manifold (M,g), vanishing theorems along with more structure on the higher cohomology groups can be obtained. For that we need to introduce the Kähler identities. Recall for $\varphi = \frac{1}{p!q!}\varphi^i_{A_p\bar B_q} dz^{A_p}\wedge d\bar z^{B_q}\otimes e_i$, $L(\varphi) = \frac{\sqrt{-1}}{p!q!}\varphi^i_{A_p\bar B_q} g_{\alpha\bar\beta} dz^\alpha\wedge d\bar z^\beta\wedge dz^{A_p}\wedge d\bar z^{B_q}\otimes e_i$. Its adjoint is defined as Λ. The proof of results below shall be done in the next set.

Proposition 0.2. *For a holomorphic vector bundle E over a Kähler manifolds, the following commutator identities hold:*

$$L\cdot\# = \#\cdot L,\ \ \Lambda\cdot\# = \#\cdot\Lambda,\ (\Lambda L - L\Lambda) = (m-p-q)\,\mathrm{id}\ \text{ on } \mathcal{A}^{p,q}(E),\ L\cdot * = *\cdot\Lambda; \tag{7}$$

$$[\Lambda,\bar\partial] = -\sqrt{-1}\partial^*, [\Lambda,\partial] = \sqrt{-1}\bar\partial^*; [L,\partial^*] = \sqrt{-1}\bar\partial, [L,\bar\partial^*] = -\sqrt{-1}\partial. \tag{8}$$

Here $[A,B] = A\cdot B - B\cdot A$. The first two can be abbreviated as $[L,\#] = 0 = [\Lambda,\#]$.

Corollary 0.2. *Under the above assumptions,*

$$\partial\partial^* - \bar\partial^*\bar\partial = \sqrt{-1}(\Lambda\partial\bar\partial - \partial\bar\partial\Lambda) = \sqrt{-1}(L\bar\partial^*\partial^* - \bar\partial^*\partial^* L); \tag{9}$$

$$\partial^*\partial - \bar\partial\bar\partial^* = \sqrt{-1}(\Lambda\bar\partial\partial - \bar\partial\partial\Lambda) = \sqrt{-1}(L\partial^*\bar\partial^* - \partial^*\bar\partial^* L); \tag{10}$$

$$\Box_\partial - \Box_{\bar\partial} = \sqrt{-1}(\Lambda e(R) - e(R)\Lambda) = \sqrt{-1}(\iota(R)\cdot L - L\cdot\iota(R)). \tag{11}$$

The key identities in Proposition 2 of last lecture are those of (8). The rest are either easy to prove or simple consequences of (8). Indeed the first one $[L, \#] = 0$ (in (7)) following from that $2\omega = \sqrt{-1} g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$ is a real $(1,1)$-form. The second one of (7) can be obtained from the first by taking adjoint. The third one is a well-known one which can be done by induction (see Huybrechts Proposition 1.2.26). The last one of (7) of L3 follows from that $\Lambda = *^{-1} \cdot L \cdot *$. This can be proved by

$$\langle x, Ly\rangle \frac{\omega^m}{m!} = x \wedge *Ly = x \wedge *2\omega \wedge y = 2\omega \wedge y \wedge *x = y \wedge * \cdot *^{-1} 2\omega * x = \langle y, \Lambda x\rangle \frac{\omega^m}{m!}.$$

For identities in (8) of L3, it suffices to prove the first set of two identities since the other two can be obtained by taking adjoint. But the first two are equivalent to each other. For example, if we assume the second identity, using $[\Lambda, \#] = 0$) we have that

$$\begin{aligned} [\Lambda, \bar{\partial}] &= [\Lambda, \#\partial\#] = \Lambda\#\partial\# - \#\partial\#\Lambda = \#[\Lambda, \partial]\# \\ &= = -\sqrt{-1}\#\bar{\partial}^*\# = \sqrt{-1} * \#\partial\# * = \sqrt{-1} * \bar{\partial} * = -\sqrt{-1}\partial^*. \end{aligned}$$

Corollary 2 can be easily derived from Proposition 2. Hence we only need to prove the second identity in (8) of the last lecture (L3).

For Kähler case we define $d : \mathcal{A}^*(E) \to \mathcal{A}^*(E)$ as $d = \partial + \bar{\partial}$, and define the corresponding Laplacian operator $\Delta_d = dd^* + d^*d = (\partial + \bar{\partial})(\partial^* + \bar{\partial}^*) + (\partial^* + \bar{\partial}^*)(\partial + \bar{\partial})$. Then (8) in L3 implies that $\Delta_d = \Box_\partial + \Box_{\bar{\partial}}$. For this we only need to check that

$$\partial\bar{\partial}^* + \bar{\partial}^*\partial = -\sqrt{-1}\left(\partial[\Lambda, \partial] + [\Lambda, \partial]\partial\right) = 0; \tag{1}$$

$$\bar{\partial}\partial^* + \partial^*\bar{\partial} = \sqrt{-1}\left(\bar{\partial}[\Lambda, \bar{\partial}] + [\Lambda, \bar{\partial}]\bar{\partial}\right) = 0. \tag{2}$$

Almost trivially we have that $[L, \bar{\partial}] = 0 = [L, \partial]$ and their adjoint. For the case E is trivial (11) of L3 implies that $\Box_\partial = \Box_{\bar{\partial}}$. Hence $\Delta_d = 2\Box_{\bar{\partial}} = 2\Box_\partial$ Together with the fact that $\mathcal{H}^{p,q}_{\bar{\partial}}(E) = \mathcal{H}^{q,p}_{\partial}(E^*)$ this implies that $h^{p,q} = h^{q,p}$. Along with the isomorphism provided by $*$ we have the Hodge diamond (cf. page 117 of [GH]).

Proposition 0.1. *On a compact Kähler manifold M. Assume that φ is a $(p,0)$-form which is holomorphic. Then $d\varphi = 0$. Similarly if $\varphi \in \mathcal{A}^{0,q}$ is ∂-closed, then $\bar{\partial}\varphi = 0$.*

Proof. It suffices to show $\partial\varphi = 0$. Using (8) of L3 and $\Lambda\partial\varphi = 0 = \bar{\partial}\varphi$,

$$(\partial\varphi, \partial\varphi) = (\varphi, \partial^*\partial\varphi) = \sqrt{-1}(\varphi, (\Lambda\bar{\partial} - \bar{\partial}\Lambda)\partial\varphi) = -\sqrt{-1}(\varphi, \bar{\partial}\Lambda\partial\varphi) = 0.$$

The other case can be reduced to this one by taking the conjugation. $\Box$

The result asserts that the holomorphicity of φ implies $\Delta_d\varphi = 0$, namely the d-harmonicity of φ. A consequence is that *the Iwasawa manifold can not be Kähler* (**Reading**). One can also derive the above result by observing that $\Box_\partial\varphi = \Box_{\bar{\partial}}\varphi = 0$. But the above proof motivates the Nakano's Lemma below, whose proof uses the computation in the above proof of the proposition (also a refinement in (5) below).

Lemma 0.1 (Nakano)**.** *Assume that $\varphi \in \mathcal{H}^{p,q}_{\bar{\partial}}(M, E)$. Then*

$$\sqrt{-1}(\Lambda e(R)\varphi, \varphi) \geq 0; \quad \sqrt{-1}(e(R)\Lambda\varphi, \varphi) \leq 0. \tag{3}$$

The equality holds in the first (second) if and only if $\partial\varphi = 0$ ($\partial^*\varphi = 0$).

Proof. That $\varphi \in \mathcal{H}^{p,q}_{\bar\partial}(M,E)$ implies that $\bar\partial\varphi = \bar\partial^*\varphi = 0$. Hence

$$\begin{aligned} 0 &\leq (\partial\varphi, \partial\varphi) = (\varphi, \partial^*\partial\varphi) = \sqrt{-1}(\varphi, (\Lambda\bar\partial - \bar\partial\Lambda)\partial\varphi) = \sqrt{-1}(\varphi, \Lambda(\bar\partial\partial + \partial\bar\partial)\varphi) \\ &= \sqrt{-1}(\varphi, \Lambda e(R)\varphi). \end{aligned}$$

Computing $(\partial^*\varphi, \partial^*\varphi)$ gives the other estimate. □

The main consequence of Proposition 2 of last lecture is the following result.

Theorem 0.1 (Kodaira)**.** *Assume that (M,g) is a Kähler manifold. Let $(\mathcal{L}, a)$ be a holomorphic line bundle. (i) Assume that its first Chern form $c_1(\mathcal{L}, a) = \frac{\sqrt{-1}}{2\pi}(-\partial\bar\partial \log a) = \frac{\sqrt{-1}}{2\pi}\Omega_{\alpha\bar\beta} dz^\alpha \wedge d\bar z^\beta$ satisfies that*

$$\Omega_{\alpha\bar\beta} + R_{\alpha\bar\beta} > 0 \tag{4}$$

where $R_{\alpha\beta} = \sum_i R^i_{i\alpha\beta}$ is the Ricci tensor of (M,g). Then $H^q(M,\mathcal{L}) = \{0\}$ for all $q \geq 1$.

(ii) If $(\mathcal{L}, a)$ is positive, then $H^q(M, \Omega^p(\mathcal{L})) = \{0\}$ for any $p + q > m$.

We prove (ii) first. As a consequence of (11) integrating on a compact M, if $\varphi \in \mathcal{H}^{p,q}_{\partial}(E)$ (if $\varphi \in \mathcal{H}^{p,q}_{\bar\partial}(E)$) we have

$$\sqrt{-1}([\Lambda, e(R)](\varphi), \varphi) = \sqrt{-1}\int_M ([\Lambda, e(R)](\varphi), \varphi)\frac{\omega^m}{m!} \leq 0 (\geq 0). \tag{5}$$

This becomes particularly useful when E is a line bundle such that its R is positive, hence can be taken as $\frac{2}{\sqrt{-1}}\omega_g$. Namely $\sqrt{-1}e(R) = L$. Now combining with (7) we have the result. The part (i) can be derived from part (ii). We shall also provide an alternate proof when we derive identities in (8) of Proposition 2 of L3.

In the rest we shall prove the second identity in (8) of last lecture. We need to develop the covariant differentiation a bit more for that. (There exists a proof via the Hodge structure see for example Wells, pages 192-194 or Huybretchs, pages 120-122.) For a Hermitian metric $g_{\alpha\bar\beta}$ on $T'M$, with respect to a local holomorphic coordinate, the connection matrix (θ^α_β) is to write $\nabla(\frac{\partial}{\partial z^\beta}) = \theta^\alpha_\beta \frac{\partial}{\partial z^\alpha}$. We define $\Gamma^\alpha_{\gamma\beta} = \theta^\alpha_\beta(\frac{\partial}{\partial z^\gamma})$. Namely $\nabla_{\frac{\partial}{\partial z^\gamma}}(\frac{\partial}{\partial z^\beta}) = \Gamma^\alpha_{\gamma\beta}\frac{\partial}{\partial z^\alpha}$. Hence for $X = X^\alpha \frac{\partial}{\partial z^\alpha}$, $\nabla_{\frac{\partial}{\partial z^\gamma}} X = \frac{\partial X^\alpha}{\partial z^\gamma}\frac{\partial}{\partial z^\alpha} + \Gamma^\beta_{\gamma\alpha}X^\alpha \frac{\partial}{\partial z^\beta}$. We shall denote the component of ∇X as $X^\alpha_{,\gamma} = \frac{\partial X^\alpha}{\partial z^\gamma} + \Gamma^\alpha_{\gamma\beta}X^\beta$. We also write $X^\alpha_{,\gamma}$ as $\nabla_\gamma X^\alpha$. This discussion also applies to a canonical connection on E. In that case for a holomorphic local frame $\{e_i\}_{1\leq i\leq r}$ and a section $s = \varphi^i e_i$ we write $D_{\frac{\partial}{\partial z^\gamma}} s = \left(\partial_\gamma \varphi^i + \Gamma^i_{\gamma j}\varphi^j\right) e_i$. Similarly denote $\partial_\gamma\varphi^i + \Gamma^i_{\gamma j}\varphi^j$ as $\varphi^i_{,\gamma}$ or $\nabla_\gamma \varphi^i$. If the meaning is clear we shall not make distinction between D and ∇. The covariant derivative on $T'M$ can be naturally extended to $T''M$ and $(T'M)^*$ and $(T''M)^*$, and the tensor products of them naturally. For example $\nabla_{\frac{\partial}{\partial z^\gamma}}(dz^\beta) = -\Gamma^\beta_{\gamma\alpha}dz^\alpha$ by our discussion on the dual bundle. Together we can make sense of covariant derivative of sections of $\mathcal{A}^{p,q}(E)$. For $\varphi \in \mathcal{A}^{p,q}(E)$, if (M,g) and (E,a) are endowed with connections, we have

$$\begin{aligned} \nabla_\gamma \varphi^i_{A_p\bar B_q} &= \partial_\gamma \varphi^i_{A_p \bar B_q} - \Gamma^s_{\gamma\alpha_1}\varphi^i_{s\alpha_2\cdots\alpha_p\bar B_q} - \cdots - \Gamma^s_{\gamma\alpha_\mu}\varphi^i_{s\alpha_1\cdots\hat\mu\cdots\alpha_p\bar B_q} \\ &\quad - \cdots - \Gamma^s_{\gamma\alpha_p}\varphi^i_{s\alpha_1\cdots\alpha_{p-1}\bar B_q} - \sum \Gamma^{\bar s}_{\gamma\bar\beta_\nu}\varphi^i_{A_p s\bar\beta_1\cdots\hat\nu\cdots\bar\beta_q} + \Gamma^i_{\gamma j}\varphi^j_{A_p\bar B_q}; \end{aligned} \tag{6}$$

$$\begin{aligned} \nabla_{\bar\gamma} \varphi^i_{A_p\bar B_q} &= \partial_{\bar\gamma}\varphi^i_{A_p\bar B_q} - \overline{\Gamma^s_{\gamma\beta_1}}\varphi^i_{A_p\bar s\bar\beta_2\cdots\bar\beta_q} - \cdots - \overline{\Gamma^s_{\gamma\beta_\nu}}\varphi^i_{A_p\bar s\bar\beta_1\cdots\hat\nu\cdots\bar\beta_q} \\ &\quad - \cdots - \overline{\Gamma^s_{\gamma\beta_q}}\varphi^i_{A_p\bar s\bar\beta_1\cdots\bar\beta_{q-1}} - \sum \Gamma^s_{\bar\gamma\alpha_\mu}\varphi^i_{s\alpha_1\cdots\hat\mu\cdots\alpha_p\bar B_q}. \end{aligned} \tag{7}$$

The canonical connection enjoys the property that $\nabla g = \nabla(g_{\alpha\bar{\beta}} dz^\alpha \otimes d\bar{z}^\beta) = 0$. In fact

$$\begin{aligned} g_{\alpha\bar{\beta},\gamma} &= \partial_\gamma g_{\alpha\bar{\beta}} - \Gamma^\delta_{\gamma\alpha} g_{\delta\bar{\beta}} = \partial_\gamma g_{\alpha\bar{\beta}} - g^{\bar{s}\delta}\partial_\gamma g_{\alpha\bar{s}}\, g_{\delta\bar{\beta}} = 0, \\ g_{\alpha\bar{\beta},\bar{\gamma}} &= \partial_{\bar{\gamma}} g_{\alpha\bar{\beta}} - \overline{\Gamma^s_{\gamma\beta}} g_{\alpha\bar{s}} = \overline{\partial_\gamma g_{\beta\bar{\alpha}} - \Gamma^s_{\gamma\beta} g_{s\bar{\alpha}}} = 0. \end{aligned}$$

This holds for the metric on E as well, without assuming the Kählerity of M. Moreover $\Gamma^{\bar{s}}_{\gamma\bar{\beta}_\nu} = \Gamma^s_{\bar{\gamma}\alpha_\mu} = 0$. The specialty of the Kähler condition is on that $\partial_\gamma g_{\alpha\bar{\beta}} = \partial_\alpha g_{\gamma\bar{\beta}}$, which is equivalent to that the Chern connection is torsion free, namely

$$\Gamma^\alpha_{\gamma\beta} = g^{\bar{s}\alpha}\partial_\gamma g_{\beta\bar{s}} = g^{\bar{s}\alpha}\partial_\beta g_{\gamma\bar{s}} = \Gamma^\alpha_{\beta\gamma}. \tag{8}$$

This symmetry is equivalent to the canonical connection being torsion free, thus being a Levi-Civita connection. From this symmetry we can check the following proposition.

Proposition 0.2. *Assume (M, g) is Kähler. For $\varphi \in \mathcal{A}^{p,q}(E)$ and a local holomorphic frame*

$$\begin{aligned} \partial\varphi &= \frac{1}{p!q!}\sum \nabla_\gamma \varphi^i_{A_p\bar{B}_q} dz^\gamma \wedge dz^{A_p} \wedge d\bar{z}^{B_q} \otimes e_i \\ &= \frac{1}{(p+1)!q!}\sum \left(\sum_{\mu=1}^{p+1} (-1)^{\mu-1} \nabla_{\alpha_\mu} \varphi^i_{\alpha_1\cdots\overset{\mu}{(\cdot)}\cdots\alpha_{p+1}\bar{B}_q}\right) dz^{A_{p+1}} \wedge d\bar{z}^{B_q} \otimes e_i; \end{aligned} \tag{9}$$

$$\begin{aligned} \bar{\partial}\varphi &= \frac{1}{p!q!}\sum \nabla_{\bar{\gamma}} \varphi^i_{A_p\bar{B}_q} d\bar{z}^\gamma \wedge dz^{A_p} \wedge d\bar{z}^{B_q} \otimes e_i \\ &= \frac{1}{p!(q+1)!}\sum (-1)^p \left(\sum_{\nu=1}^{q+1} (-1)^{\nu-1} \nabla_{\bar{\beta}_\nu} \varphi^i_{A_p\bar{\beta}_1\cdots\overset{\nu}{(\cdot)}\cdots\bar{\beta}_{q+1}}\right) dz^{A_p} \wedge d\bar{z}^{B_{q+1}} \otimes e_i. \end{aligned} \tag{10}$$

Now we prove identities in Proposition 2 of the last lecture. The corresponding version without Kähler condition can be found in Ma-Marinescu's book *Holomorphic Morse Inequality and Bergman Kernels*. For this we first need the following formula on $\Lambda(\varphi)$ for $\varphi = \frac{1}{p!q!}\sum \varphi_{A_p\bar{B}_q} dz^{A_p} \wedge d\bar{z}^{B_q}$,

$$\Lambda\varphi = \frac{1}{(p-1)!(q-1)!}\frac{(-1)^{p-1}}{\sqrt{-1}}\sum g^{\bar{\beta}\alpha}\varphi_{\alpha A_{p-1}\bar{\beta}\bar{B}_{q-1}} dz^{A_{p-1}} \wedge d\bar{z}^{B_{q-1}}. \tag{11}$$

One simply needs to check that the formula holds for singletons. We need a formula of $\bar{\partial}^*$ in terms of the covariant derivatives for the second identity of (8). Note the lemma below.

Lemma 0.2. *For Kähler manifold, with $g = \det(g_{\alpha\bar{\beta}})$*

$$\sum_\beta \Gamma^\beta_{\gamma\beta} = \partial_\gamma \log g.$$

Combining this with the equation (5) of last lecture we have the result below.

Proposition 0.3. *On a Kähler manifold (M, g) and holomorphic vector bundle E,*

$$(\bar{\partial}^*\varphi)^{i\bar{\Lambda}_p N_q} = (-1)^{p+1}\sum_{\gamma=1}^m \nabla_\gamma \varphi^{i\bar{\Lambda}_p\gamma N_q};\; (\bar{\partial}^*\varphi)^i_{A_p\bar{B}_q} = (-1)^{p+1}\sum_{\gamma=1}^m \nabla_\gamma\left(g^{\bar{\delta}\gamma}\varphi^i_{A_p\bar{\delta}\bar{B}_q}\right). \tag{12}$$

Proof. By the definition we have that

$$\begin{aligned}\sum_{\gamma=1}^{m}\nabla_\gamma\varphi^{i\bar\Lambda_p\gamma N_q} &= \partial_\gamma\varphi^{i\bar\Lambda_p\gamma N_q}+\Gamma^i_{\gamma j}\varphi^{j\bar\Lambda_p\gamma N_q}+\Gamma^\gamma_{\gamma s}\varphi^{i\bar\Lambda_p s N_q}+\sum_{\nu=1}^{q}\Gamma^{\mu_\nu}_{\gamma s}\varphi^{i\bar\Lambda_p\gamma\mu_1\cdots\overset{\nu}{(s)}\cdots\mu_q}\\ &= \partial_\gamma\varphi^{i\bar\Lambda_p\gamma N_q}+\Gamma^i_{\gamma j}\varphi^{j\bar\Lambda_p\gamma N_q}+\Gamma^\gamma_{\gamma s}\varphi^{i\bar\Lambda_p s N_q}.\end{aligned}$$

Now the result follows from (5) of last lecture and the above lemma. □

Now by the equations (11) and (9) we have that

$$\begin{aligned}((\Lambda\partial \quad - \quad \partial\Lambda)\varphi)^i_{A_p\bar B_{q-1}} &= \frac{(-1)^p}{\sqrt{-1}}\sum_{\alpha,\beta=1}^{m}g^{\bar\beta\alpha}(\partial\varphi)^i_{\alpha A_p\bar\beta\bar B_{q-1}}\\ &\quad-\left(\sum_{\mu=1}^{p}(-1)^{\mu-1}\nabla_{\alpha_\mu}(\Lambda\varphi)^i_{\alpha_1\cdots\overset{\mu}{(\cdot)}\cdots\alpha_p\bar B_{q-1}}\right)\\ &= \frac{(-1)^p}{\sqrt{-1}}\sum_{\alpha,\beta}g^{\bar\beta\alpha}\left(\nabla_\alpha\varphi^i_{A_p\bar\beta\bar B_{q-1}}+\sum_{\mu=1}^{p}(-1)^{\mu}\nabla_{\alpha_\mu}\varphi^i_{\alpha\alpha_1\cdots\overset{\mu}{(\cdot)}\cdots\alpha_p\bar\beta\bar B_{q-1}}\right)\\ &\quad-\sum_{\mu=1}^{p}(-1)^{\mu-1}\frac{(-1)^{p-1}}{\sqrt{-1}}\left(\nabla_{\alpha_\mu}g^{\bar\beta\alpha}\varphi^i_{\alpha\alpha_1\cdots\overset{\mu}{(\cdot)}\cdots\alpha_p\bar\beta\bar B_{q-1}}\right)\\ &= \sqrt{-1}(\bar\partial^*\varphi)^i_{A_p\bar B_{q-1}}.\end{aligned}$$

The last equation uses (12). This proves the second equation in (8) of last lecture.

The alternate proof of the Kodaira's vanishing theorem (i) involves another approach exploiting Bochner's idea. It involves comparing $\Box_{\bar\partial}$ with twice co-variant derivatives. The proof is direct calculations.

Theorem 0.2. *For Kähler manifold (M,g), a holomorphic vector bundle (E,a) with canonical connections, and a $\varphi\in\mathcal{A}^{p,q}(E)$,*

$$\begin{aligned}(\Box_{\bar\partial}\varphi)^i_{A_p\bar B_q} &= -\sum_{\alpha,\beta=1}^{m}g^{\bar\beta\alpha}\nabla_\alpha\nabla_{\bar\beta}\varphi^i_{A_p\bar B_q}+\sum_{\nu=1}^{q}\Omega^{i\ \bar\tau}_{j\ \bar\beta_\nu}\varphi^j_{A_p\bar\beta_1\cdots\overset{\nu}{(\bar\tau)}\cdots\bar\beta_q}+\sum_{\nu=1}^{q}R^{\bar\tau}_{\bar\beta_\nu}\varphi^i_{A_p\bar\beta_1\cdots\overset{\nu}{(\bar\tau)}\cdots\bar\beta_q}\\ &\quad-\sum_{\mu=1}^{p}\sum_{\nu=1}^{q}R_{\alpha_\mu}{}^{\sigma\bar\tau}{}_{\bar\beta_\nu}\varphi^i_{\alpha_1\cdots\overset{\mu}{(\sigma)}\cdots\alpha_p\bar\beta_1\cdots\overset{\nu}{(\bar\tau)}\cdots\bar\beta_q};\end{aligned}\tag{13}$$

$$\begin{aligned}(\Box_{\bar\partial}\varphi)^i_{A_p\bar B_q} &= -\sum_{\alpha,\beta=1}^{m}g^{\bar\beta\alpha}\nabla_{\bar\beta}\nabla_\alpha\varphi^i_{A_p\bar B_q}+\sum_{\nu=1}^{q}\Omega^{i\ \bar\tau}_{j\ \bar\beta_\nu}\varphi^j_{A_p\bar\beta_1\cdots\overset{\nu}{(\bar\tau)}\cdots\bar\beta_q}+\sum_{\mu=1}^{p}R^{\sigma}_{\alpha_\mu}\varphi^i_{\alpha_1\cdots\overset{\mu}{(\sigma)}\cdots\alpha_p\bar B_q}\\ &\quad-\sum_{j=1}^{r}\Omega^i_j\varphi^j_{A_p\bar B_q}-\sum_{\mu=1}^{p}\sum_{\nu=1}^{q}R_{\alpha_\mu}{}^{\sigma\bar\tau}{}_{\bar\beta_\nu}\varphi^i_{\alpha_1\cdots\overset{\mu}{(\sigma)}\cdots\alpha_p\bar\beta_1\cdots\overset{\nu}{(\bar\tau)}\cdots\bar\beta_q}.\end{aligned}\tag{14}$$

Integration by part of (13) gives another proof of part (i) of Kodaira's theorem.

Lecture 5 – Local Rigidity and Strong Rigidity of Kähler Manifolds by L. Ni

The results here concerns a class of Kähler manifolds which is non-positively curved. Besides the hyperbolic space forms (which has constant holomorphic sectional curvature -1), these classes of manifolds include Hermitian symmetric spaces of noncompact type and the so called Cartan domains. They are all Kähler manifolds and algebraic. The comprehensive results require some tedious, but accurate computations of curvature tensors of these spaces built upon the Lie theory. Our discussions focus on the simplest case without it.

Theorem 0.1 (Calabi-Vesentini). *Assume that M^m $(m \geq 2)$ is a compact quotient of the Hermitian symmetric spaces of noncompact type. Then $H^1(M, T'M) = \{0\}$. In particular this implies that such a manifold is locally rigid.*

Here we need some notations/results to better understand the above result. The *local rigidity* is related to the concept of *the deformation of complex structures*, which by now is a huge subject. For the case of the Riemann surfaces it was studied by Riemann and later was developed into a subject named *Teichmüller theory* after its most significant contributor. The high dimensional version was originated and developed by Kodaira and Spencer in a series of papers in 1950s. The book of Morrow-Kodaira contains a good coverage on this.

Definition 0.2. *Let B be a connected complex manifold and let $\{M_t\}_{t\in B}$ be a set of compact complex manifolds depending on t. We say M_t depends on t holomorphically if there is a complex manifold $\mathcal{M}$ and a map $\pi : \mathcal{M} \to B$ such that (1) $\pi^{-1}(t) = M_t$ for each $t \in B$; (2) the rank of π is $\dim_{\mathbb{C}} B$.*

This M_t is then called a *complex analytic family.* A such family with B being a small ball is called a local deformation of M_0. It is not too hard to prove that M_t and $M_{t'}$ are diffeomorphic to each other. We say that M_0 is locally rigid if for any complex family, there exists a small neighborhood U of 0 such that $\pi^{-1}(U) = U \times M_0$ holomorphically. The following result of Frölicher-Nijenhuis connects the local rigidity with $H^1(M, T'M)$.

Theorem 0.3. *Let $(\mathcal{M}, M_t, B, \pi)$ be a complex family $(\mathcal{M}, M_t, B, \pi)$ as above. Assume that $H^1(M_0, T'M_0) = \{0\}$. Then M_0 is locally rigid.*

Theorem of CV is weaker than the statement that*if M' is a complex manifold diffeomorphic to M_0, a compact quotient of a Hermitian symmetric space, then it is biholomorphic to M_0.* However the result contrasts sharply with the case $m = 1$ since a Riemann surface of genus $g \geq 2$ (covered by the unit ball) has nontrivial analytic deformations.

Using a nonlinear analogue of the techniques of Calabi-Vesentini, Siu proved a strong generalization below in Kähler category, built upon the harmonic maps $u : M \to N$ with M being compact and N with sectional curvature $K^N \leq 0$, produced by Eells and Sampson (1964). Siu's work was motivated by a similar rigidity result of Mostow earlier.

Theorem 0.4 (Siu). *Assume that (M^m, g) $(m \geq 2)$ is a compact Kähler manifold. Assume further that it is homotopically equivalent to a compact quotient N of a Hermitian symmetric spaces of noncompact type. Then M and N are biholomorphic to each other.*

In this regards we should mention a result Kodaira-Spencer, whose proof we refer you to the book of Morrow-Kodaira, appealing to results about the fourth order elliptic PDEs.

Theorem 0.5. *Let $(\mathcal{M}, M_t, B, \pi)$ be a complex family $(\mathcal{M}, M_t, B, \pi)$ as above. Assume that M_0 is Kähler. Then there exists a small neighborhood U of 0 such that M_t is also Kähler for $t \in U$.*

Since the Kählerity may fail to hold for M_t with t being too far away from 0, it remains interesting to prove Siu's result without assuming manifold M being Kähler, even under the assumption that M is diffeomorphic to N.

In the rest we prove the results of Calabi-Vesentini and Siu. We first prove a relevant vanishing theorem due to Gigante and Girbau (generalizing Kodaira's part (ii)).

Theorem 0.6. *Assume that (M, g) is Kähler and $(\mathcal{L}, a)$ is a holomorphic line bundle such that its first Chern form $c_1(\mathcal{L}, a) \leq 0$ with rank k. Then $H^q(M, \Omega^p(\mathcal{L})) = \{0\}$ for any $p + q \leq k - 1$.*

Proof. The proof uses the same idea as part (ii) of Kodaira's vanishing theorem. Now we simply compute out $\sqrt{-1}e(R)\Lambda\varphi$ and $\sqrt{-1}\Lambda e(R)\varphi$:

$$
\begin{aligned}
\left(\Lambda \cdot \sqrt{-1}e(R)(\varphi)\right)_{A_p\bar{B}_q} &= \Omega^i_j \varphi^j_{A_p\bar{B}_q} - \sum_{\alpha,\beta=1}^{m} g^{\bar{\beta}\alpha} \sum_{\mu=1}^{p} \Omega^i_{j\,\alpha_\mu\bar{\beta}} \varphi^j_{\alpha_1\cdots\overset{\mu}{(\alpha)}\cdots\alpha_p\bar{B}_q} \\
&\quad - \sum_{\alpha,\beta=1}^{m} g^{\bar{\beta}\alpha} \sum_{\nu=1}^{q} \Omega^i_{j\,\alpha\bar{\beta}_\nu} \varphi^j_{A_p\bar{\beta}_1\cdots\overset{\nu}{(\bar{\beta})}\cdots\bar{\beta}_q} \qquad (1) \\
&\quad + \sum_{\alpha,\beta=1}^{m} g^{\bar{\beta}\alpha} \sum_{\mu=1}^{p}\sum_{\nu=1}^{q} \Omega^i_{j\,\alpha_\mu\bar{\beta}_\nu} \varphi^j_{\alpha_1\cdots\overset{\mu}{(\alpha)}\cdots\alpha_p\bar{\beta}_1\cdots\overset{\nu}{(\bar{\beta})}\cdots\bar{\beta}_q};
\end{aligned}
$$

$$
\left(\sqrt{-1}e(R) \cdot \Lambda(\varphi)\right)_{A_p\bar{B}_q} = \sum_{\alpha,\beta=1}^{m} g^{\bar{\beta}\alpha} \sum_{\mu=1}^{p}\sum_{\nu=1}^{q} \Omega^i_{j\,\alpha_\mu\bar{\beta}_\nu} \varphi^j_{\alpha_1\cdots\overset{\mu}{(\alpha)}\cdots\alpha_p\bar{\beta}_1\cdots\overset{\nu}{(\bar{\beta})}\cdots\bar{\beta}_q}. \qquad (2)
$$

The result follows by applying the above to the special case of line bundle, and working with the Kähler metric g' whose Kähler form is $\omega' = \epsilon\omega - \frac{\sqrt{-1}}{2}\Omega_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$ and taking $\epsilon \to 0$, since $\Omega \to -k$ and the other two positive terms can at the best contribute $p + q < k$. □

For Calabi-Vesentini's theorem we focus on the special case that M is a compact quotient of complex hyperbolic space. The curvature tensor $R_{\alpha\bar{\beta}\gamma\bar{\delta}}$ for this manifold is $-(g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + g_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}})$. To prove the result it suffices to show that any harmonic $(0,1)$-form valued in $T'M$ must vanish. As in the above result the negativity of the curvature will help. Direct computation shows that $([\Lambda, \sqrt{-1}e(R)]\varphi, \varphi) < 0$ for $\varphi \in \mathcal{H}^{(0,1)}_{\bar{\partial}}(M, T'M)$. We provide some useful details below for general Kähler manifolds.

First the curvature Ω (for $T'M$) enjoys a lot of symmetry for Kähler manifold, Since $g_{\alpha\bar{\beta}}$ is the complex Hessian of a real function f, namely $g_{\alpha\bar{\beta}} = \frac{\partial^2 f}{\partial z^\alpha \partial \bar{z}^\beta}$. Direct calculation then shows that

$$
\Omega_{i\bar{l}\alpha\bar{\beta}} = -\left(\frac{\partial^4 f}{\partial z^\alpha \partial z^i \partial \bar{z}^\beta \partial \bar{z}^l} - g^{\bar{k}j} \frac{\partial^3 f}{\partial z^k \partial \bar{z}^l \partial \bar{z}^\beta} \frac{\partial^3 f}{\partial z^\alpha \partial z^i \partial \bar{z}^j}\right).
$$

From this it is easy to see that $\Omega_{i\bar{l}\alpha\bar{\beta}}$ is symmetric in i, α and j, l. Hence we may define $Q : S^2(T'M) \to S^2(T'M)$ as $Q(\xi^{i\alpha} \cdot \frac{1}{2}(e_i \otimes e_\alpha + e_\alpha \otimes e_i)) \doteqdot \Omega_{i\bar{l}\alpha\bar{\beta}}\xi^{i\alpha} \cdot \frac{1}{2}(e_l \otimes e_\beta + e_\beta \otimes e_l)$. Here we use a unitary frame. Then $Q : S^2(T'M) \to S^2(T'M)$ is a Hermitian symmetric transformation with $Q(\xi,\bar\xi)=\Omega_{i\bar l\alpha\bar\beta}\xi^{i\alpha}\overline{\xi^{l\beta}}$. Applying (1) and (2), at a point, and using a unitary frame at the point, for $\varphi\in\mathcal{A}^{0,q}(T'M)$,

$$([\Lambda,\sqrt{-1}e(R)]\varphi,\varphi)=\Omega_{j\bar i}\varphi_{\bar j\bar B_q}\overline{\varphi_{\bar i\bar B_q}}-\sum_{\nu=1}^{q}\Omega_{j\bar i\alpha\bar\beta_\nu}\varphi_{\bar j\bar\beta_1\cdots\overset{\nu}{(\bar\alpha)}\cdots\bar\beta_q}\overline{\varphi_{\bar i\bar B_q}},\ \text{with } \varphi_{\bar j\bar B_q}=g_{i\bar j}\varphi^i_{\bar B_q}.$$

Let $\xi^{j\alpha}_{\bar B^{\hat\nu}_q}\doteqdot\varphi_{\bar j\bar\beta_1\cdots\overset{\nu}{(\bar\alpha)}\cdots\bar\beta_q}$. The second term above can be written as $\sum_{\nu=1}^q\Omega_{j\bar i\alpha\bar\beta_\nu}\xi^{j\alpha}_{\bar B^{\hat\nu}_q}\overline{\xi^{i\beta_\nu}_{\bar B^{\hat\nu}_q}}$. For the compact quotient of a hyperbolic space form it can be checked that

$$\Omega_{j\bar i}\varphi^j_{\bar B_q}\overline{\varphi^i_{\bar B_q}}=-(m+1)|\varphi^j_{\bar B_q}|^2,\quad \sum_{\nu=1}^q\Omega_{j\bar i\alpha\bar\beta_\nu}\xi^{j\alpha}_{\bar B^{\hat\nu}_q}\overline{\xi^{i\beta_\nu}_{\bar B^{\hat\nu}_q}}\le-2|\varphi^j_{\bar B_q}|^2.$$

This together with Nakano's Lemma implies that if $m\ge 2$, $H^q(M,T'M)=\{0\}$ for all $q\ge 1$.

For Siu's theorem we first introduce the concept of harmonic maps. A map $u:(M,g)\to(N,h)$ is called harmonic if it is the critical point of the energy functional

$$\mathcal{E}(u)=\frac{1}{2}\int_M|df|^2\,d\mu_M.$$

If $(z^1,\cdots,z^m)$ and $(w^1,\cdots,w^n)$ are the local coordinates of M and N, then $df(\frac{\partial}{\partial z^\alpha})=f^i_\alpha\frac{\partial}{\partial w^i}+f^{\bar j}_\alpha\frac{\partial}{\partial\bar w^j}\doteqdot\partial f(\frac{\partial}{\partial z^\alpha})+\partial\bar f(\frac{\partial}{\partial z^\alpha})$. Similarly we also write $df(\frac{\partial}{\partial\bar z^\alpha})=\bar\partial f(\frac{\partial}{\partial\bar z^\alpha})+\bar\partial\bar f(\frac{\partial}{\partial\bar z^\alpha})$ with $\bar\partial f(\frac{\partial}{\partial\bar z^\alpha})=f^i_{\bar\alpha}\frac{\partial}{\partial w^i}$ and $\bar\partial\bar f(\frac{\partial}{\partial\bar z^\alpha})=f^{\bar j}_{\bar\alpha}\frac{\partial}{\partial\bar w^j}$. Hence

$$|df|^2=g^{\bar\beta\alpha}h_{i\bar j}\left(f^i_\alpha\overline{f^j_\beta}+f^{\bar j}_\alpha\overline{f^{\bar i}_\beta}\right)\doteqdot|\partial f|^2+|\partial\bar f|^2.$$

Eells-Sampson proved that if N is a Riemannian manifold with *non-positive sectional curvature*, then any initial smooth map $u_0:M\to N$ can be deformed into a harmonic map u in the same homotopy class. The main step of Siu's theorem is to prove that such a harmonic homotopically equivalent map must be a biholomorphism.

Using the terminology of pull-back bundle $\partial f=f^i_\alpha dz^\alpha\otimes\frac{\partial}{\partial w^i}$ can be viewed as a section of $\mathcal{A}^{1,0}(f^*T'N)$. The pull-back bundle $E=f^*T'N$ usually does not endow a holomorphic structure except some very special situations. Similarly $\bar\partial f=f^i_{\bar\alpha}d\bar z^\alpha\otimes\frac{\partial}{\partial w^i}$ is a section of $\mathcal{A}^{0,1}(E)$. A map is harmonic if and only if df viewed as $\mathcal{A}^1(f^*TN)$ is a harmonic 1-form. A map is holomorphic if $\bar\partial f\equiv 0$. In this sense, the result of Siu is a nonlinear vanishing theorem for "harmonic" $(0,1)$-forms. The issue is nonlinear since the harmonicity depends on the metric on $f^*T'N$, which involves f. The holomorphicity of a harmonic map is related to the notion of the *complex sectional curvature*. For a Kähler manifold (M,g), the nonpositivity of the complex sectional curvature means that

$$(*)\qquad R_{i\bar j s\bar t}(a^i\bar b^j-c^i\bar d^j)\overline{(a^t\bar b^s-c^t\bar d^s)}\le 0$$

for any complex vector $\vec a(=(a^1,\cdots,a^m))$, $\vec b$, $\vec c$ and $\vec d$. It is originally called that (M,g) has *strongly non-positive sectional curvature in the sense of Siu*, which is in general stronger than the non-positivity of the sectional curvature (or bisectional curvature). On a Kähler manifold, if $\{E_i\}$ is a unitary basis of $T'M$, and letting $X=a^iE_i$, $Y=b^iE_i$, $Z=d^iE_i$ and $W=c^iE_i$, then (*) is equivalent to

$$\langle\mathrm{Rm}(U\wedge V),\overline{U\wedge V}\rangle=\langle\mathrm{Rm}((X+\bar Z)\wedge(\bar Y+W)),\overline{(X+\bar Z)\wedge(\bar Y+W)}\rangle\le 0,$$

Here $U = X + \bar{Z}, V = \bar{Y} + W$, Rm denotes the complexified curvature operator. We say that the curvature is *strongly negative* if the expression is negative when at least for one pair (i, j), $a^i \bar{b}^j - c^i \bar{d}^j \neq 0$. It can be checked easily that on a complex hyperbolic space the strong negativity holds. Siu proved the holomorphicity of a harmonic map via a $\partial\bar{\partial}$-lemma. We use an argument of Sampson, which uses the Hopf's differential:

$$\phi = \phi_{\alpha\beta} dz^\alpha \otimes dz^\beta = h_{i\bar{j}} f^i_\alpha f^{\bar{j}}_{\bar{\beta}} dz^\alpha \otimes d\bar{z}^\beta.$$

We define $\operatorname{div}(\phi) \doteqdot g^{\bar{\gamma}\beta}\phi_{\alpha\beta,\bar{\gamma}} dz^\alpha$. This is only formally related to ∂^* acting on $\mathcal{A}^{p,q}(E)$

$$(\partial^* \varphi)^i_{A_{p-1}\bar{B}_q} = -\sum g^{\bar{\beta}\alpha} \nabla_{\bar{\beta}} \phi^i_{\alpha A_{p-1}\bar{B}_q}. \tag{3}$$

Let $\Gamma^\alpha_{\gamma\beta}$ and Γ^i_{jk} be the connection coefficients of M and N. We denote $f^i_{\alpha|\beta} = f^i_{\alpha,\beta} + \Gamma^i_{kl} f^k_\alpha f^l_\beta$, with $f^i_{\alpha,\beta} = \partial_\beta f^i_\alpha - \Gamma^s_{\beta\alpha} f^i_s$. In terms of the invariant notation $D'\partial f(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial z^\beta}) = f^i_{\alpha|\beta} \frac{\partial}{\partial w^i}$. Similarly $D''\partial f = f^i_{\alpha|\bar{\beta}} \frac{\partial}{\partial w^i}$ with $f^i_{\alpha|\bar{\beta}} = \partial_{\bar{\beta}} f^i_\alpha + \Gamma^i_{kl} f^k_\alpha f^l_{\bar{\beta}}$. The harmonic map equation, as the Euler-Lagrange equation of the energy $\mathcal{E}(f)$ has a simple form

$$g^{\bar{\beta}\alpha} f^i_{\alpha|\bar{\beta}} = 0, \qquad \text{for all } 1 \le i \le n = \dim(N). \tag{4}$$

The proof is based on the observation that $(1, \partial^*\sigma) = 0$ if $\sigma = \operatorname{div}(\phi)$. Adapting the covariant derivative to the tensors $f^i_{\alpha|\bar{\beta}} dz^\alpha \otimes d\bar{z}^\beta \frac{\partial}{\partial w^i}$ we have

$$f^i_{\alpha|\bar{\beta}|\bar{\gamma}} = \partial_{\bar{\gamma}} f^i_{\alpha|\bar{\beta}} - \Gamma^{\bar{\delta}}_{\bar{\beta}\bar{\gamma}} f^i_{\alpha|\bar{\delta}} + \Gamma^i_{kj} f^k_{\alpha|\bar{\beta}} f^j_{\bar{\gamma}}; \tag{5}$$

$$f^i_{\alpha|\bar{\gamma}|\bar{\beta}} = \partial_{\bar{\beta}} f^i_{\alpha|\bar{\gamma}} - \Gamma^{\bar{\delta}}_{\bar{\beta}\bar{\gamma}} f^i_{\alpha|\bar{\delta}} + \Gamma^i_{kj} f^k_{\alpha|\bar{\gamma}} f^j_{\bar{\beta}}; \tag{6}$$

$$f^i_{\alpha|\bar{\beta}|\bar{\gamma}} - f^i_{\alpha|\bar{\gamma}|\bar{\beta}} = -R^i_{st\bar{j}} \left(f^t_{\bar{\beta}} f^{\bar{j}}_{\bar{\gamma}} - f^t_{\bar{\gamma}} f^{\bar{j}}_{\bar{\beta}} \right) f^s_\alpha. \tag{7}$$

The first two are definitions, the third one is the commutator formula which can be obtained from the first two by computation. The proof of the holomorphicity is based on the two equations below:

$$\sigma = \operatorname{div}(\phi) = g^{\bar{\gamma}\beta} h_{i\bar{j}} \left(f^i_{\alpha|\bar{\gamma}} f^{\bar{j}}_{\bar{\beta}} + f^i_\alpha f^{\bar{j}}_{\bar{\beta}|\bar{\gamma}} \right) dz^\alpha = g^{\bar{\gamma}\beta} h_{i\bar{j}} f^i_{\alpha|\bar{\gamma}} f^{\bar{j}}_{\bar{\beta}} dz^\alpha; \tag{8}$$

$$\begin{aligned} -\partial^*\sigma &= g^{\bar{\beta}\alpha} g^{\bar{\eta}\delta} h_{i\bar{j}} \left(f^i_{\alpha|\bar{\eta}} f^{\bar{j}}_{\bar{\delta}|\bar{\beta}} + f^i_{\alpha|\bar{\eta}|\bar{\beta}} f^{\bar{j}}_{\bar{\delta}} \right) = g^{\bar{\beta}\alpha} g^{\bar{\eta}\delta} h_{i\bar{j}} f^i_{\alpha|\bar{\eta}} f^{\bar{j}}_{\bar{\delta}|\bar{\beta}} \\ &\quad - g^{\bar{\beta}\alpha} g^{\bar{\eta}\delta} h_{i\bar{j}} \left(R^i_{st\bar{l}} \left(f^t_{\bar{\eta}} f^{\bar{l}}_{\bar{\beta}} - f^t_{\bar{\beta}} f^{\bar{l}}_{\bar{\eta}} \right) f^s_\alpha f^{\bar{j}}_{\bar{\delta}} \right) = I + II. \end{aligned} \tag{9}$$

In the case $m = 1$ (8) implies that ϕ is holomorphic, which is very useful in Teichmüller theory. To prove the result we observe that $I \geq 0$. Under the assumption of the curvature being strongly negative we deduce that $II \geq 0$ and $II = 0$ forces that the map is either holomorphic or antiholomorphic. Under the unitary frames we have

$$\begin{aligned} II &= -R_{s\bar{j}t\bar{l}} \left(f^t_{\bar{\eta}} f^{\bar{l}}_{\bar{\alpha}} - f^t_{\bar{\alpha}} f^{\bar{l}}_{\bar{\eta}} \right) f^s_\alpha f^{\bar{j}}_\eta \overset{\alpha \leftrightarrow \eta}{=} R_{s\bar{j}t\bar{l}} \left(f^t_{\bar{\eta}} f^{\bar{l}}_{\bar{\alpha}} - f^t_{\bar{\alpha}} f^{\bar{l}}_{\bar{\eta}} \right) f^s_\eta f^{\bar{j}}_\alpha \\ &= -\frac{1}{2} R_{s\bar{j}t\bar{l}} \left(f^t_{\bar{\eta}} f^{\bar{l}}_{\bar{\alpha}} - f^t_{\bar{\alpha}} f^{\bar{l}}_{\bar{\eta}} \right) \left(f^s_\alpha f^{\bar{j}}_\eta - f^s_\eta f^{\bar{j}}_\alpha \right) = -\frac{1}{2} R_{s\bar{j}t\bar{l}} \overline{\left(f^{\bar{t}}_\eta f^l_\alpha - f^{\bar{t}}_\alpha f^l_\eta \right)} \left(f^s_\alpha f^{\bar{j}}_\eta - f^s_\eta f^{\bar{j}}_\alpha \right). \end{aligned}$$

The claimed result follows by analyzing the above expression with $R_{s\bar{j}t\bar{l}} = g_{s\bar{j}} g_{t\bar{l}} + g_{s\bar{l}} g_{t\bar{j}}$.

Lecture 6 –The L^2-$\bar\partial$-estimate and Complex Frobenius Theorems by L. Ni

We group them together since they are all about solving the $\bar\partial$-equation. For the simplicity we limit the discussion for the L^2-estimate to line bundles over compact Kähler manifolds. The scope can be much larger including noncompact complex manifolds and vector bundles.

The proof of Gigante-Girbau theorem provides the following formula for line bundle $(\mathcal{L}, a)$:

$$\int_M \|\bar\partial\varphi\|^2 + \|\bar\partial^*\varphi\| = \int_M -\Omega\cdot a\cdot \varphi_{A_p\bar B_q}\overline{\varphi^{\bar A_p B_q}} + g^{\bar\beta\alpha} a\sum_{\mu=1}^{p}\Omega_{\alpha_\mu\bar\beta}\varphi_{\alpha_1\cdots\overset{\mu}{(\alpha)}\cdots\alpha_p\bar B_q}\overline{\varphi^{\bar A_p B_q}}$$

$$(1)\qquad +\int_M \sum_{\alpha,\beta=1}^{m} g^{\bar\beta\alpha}a\sum_{\nu=1}^{q}\Omega_{\alpha\bar\beta_\nu}\varphi_{A_p\bar\beta_1\cdots\overset{\nu}{(\bar\beta)}\cdots\bar\beta_q}\overline{\varphi^{\bar A_p B_q}} + \int_M \|\partial\varphi\|^2 + \|\partial^*\varphi\|^2.$$

If we pick a coordinate (near a point) so that $g_{\alpha\bar\beta} = \delta_{\alpha\beta}$ and $\Omega_{\alpha\bar\beta}$ is diagonalized with eigenvalues $\{\lambda_\gamma\}$ (at the point), and if we assume that

$$(2)\qquad \min_{x\in M}\left(\sum_{i=1}^{p}\lambda_{\gamma_i} + \sum_{j=1}^{q}\lambda_{\gamma_j} - \sum_{\gamma=1}^{m}\lambda_\gamma\right) \ge c > 0$$

(everywhere on M), then (1) implies

$$(*)\qquad \int_M \|\bar\partial\varphi\|^2 + \|\bar\partial^*\varphi\| \ge \int_M \|\partial\varphi\|^2 + \|\partial^*\varphi\| + c\int_M \|\varphi\|^2 \ge c\int_M \|\varphi\|^2 \qquad .$$

If $\Box_{\bar\partial}\varphi = 0$ the estimate (*) implies $\varphi = 0$. We shall see that this also is the key of the L^2-estimate. The set-up is to consider $\bar\partial : \mathcal{D}(\bar\partial) \subset L^2_{p,q}(\mathcal{L}) \to L^2_{p,q+1}(\mathcal{L})$, where $\mathcal{D}(\bar\partial) = \{\varphi \in L^2_{p,q}(\mathcal{L}) | \bar\partial\varphi \in L^2_{p,q+1}(\mathcal{L})\}$. For any $\varphi \in L^2_{p,q}(\mathcal{L})$, $\bar\partial\varphi$ is understood in the sense of distribution. Clearly $\mathcal{A}^{p,q}(\mathcal{L}) \in \mathcal{D}(\bar\partial)$. It can be checked that $\bar\partial$ is closed. Namely if $x_n \to x$ and $\bar\partial x_n \to y$ we have $\bar\partial x = y$, since $\forall\varphi \in \mathcal{A}^{p,q}(\mathcal{L})$, first $(\bar\partial x_n, \varphi) \to (y, \varphi)$, at the mean time $(\bar\partial x_n, \varphi) = (x_n, \bar\partial^*\varphi) \to (x, \bar\partial^*\varphi) = (\bar\partial x, \varphi)$. Now define $y^* \doteqdot \bar\partial^* y$, for $y \in \mathcal{D}(\bar\partial^*)$ with

$$\mathcal{D}(\bar\partial^*) \doteqdot \{y \in L^2_{p,q+1}(\mathcal{L}) \,|\, \exists\, y^* \in L^{p,q}_2(\mathcal{L}),\text{ such that } (\bar\partial x, y) = (x, y^*), \forall\, x \in \mathcal{D}(\bar\partial)\}.$$

It is well defined since $\mathcal{A}^{p,q}(\mathcal{L})$ is dense in $L^2_{p,q}(\mathcal{L})$. It can be seen that $\bar\partial^*$ is also closed. In fact, if $y_n \in \mathcal{D}(\bar\partial^*)$ and $y_n \to y$, $\bar\partial^* y_n \to y^*$, $\forall\varphi \in \mathcal{A}^{p,q}$, $(\varphi, \bar\partial^* y_n) = (\bar\partial\varphi, y_n) \to (\bar\partial\varphi, y)$. At the mean time $(\varphi, \bar\partial^* y_n) \to (\varphi, y^*)$. This implies that $(\bar\partial\varphi, y) = (\varphi, y^*)$ for all $\varphi \in \mathcal{A}^{p,q}$. Hence $y \in \mathcal{D}(\bar\partial^*)$ and $y^* = \bar\partial^* y$.

Lemma 0.1.
$$L^2_{p,q}(\mathcal{L}) = \ker\bar\partial \oplus (\ker\bar\partial)^\perp; \quad (\ker\bar\partial)^\perp \subset \ker\bar\partial^*.$$

Proof. First note that $\forall\psi \in \mathcal{A}^{p,q-1}$ $\bar\partial\psi \in \ker\bar\partial$. Hence $\forall\varphi \in (\ker\bar\partial)^\perp$, $(\varphi, \bar\partial\psi) = 0 = (0, \psi)$. This proves that $\bar\partial^*\varphi = 0$. □

The main theorem of the L^2-$\bar\partial$-estimate is the following:

Theorem 0.1. *If (*) holds then for any $\varphi \in \mathcal{D}(\bar\partial)$ with $\bar\partial\varphi = 0$, there exists a $\psi \in L^2_{p,q-1}(\mathcal{L})$ such that $\bar\partial\psi = \varphi$. Moreover*

$$(3)\qquad \int_M \|\psi\|^2 \le \frac{1}{c}\int_M \|\varphi\|^2.$$

Proof. First we derive an estimate using (*). For any $v \in L^2_{p,q}(\mathcal{L})$ write it as v_1+v_2 according to the above decomposition. Since $v_2 \in (\ker \bar\partial)^{\perp}$ and $\varphi \in \ker \bar\partial$,

$$|(\varphi, v)| = |(\varphi, v_1) + (\varphi, v_2)| = |(\varphi, v_1)| \leq \left(\frac{1}{c}\|\varphi\|^2 \cdot c\|v_1\|^2\right)^{\frac{1}{2}}.$$

Combining with (*), which also holds for $v_1 \in \mathcal{D}(\bar\partial) \cap \mathcal{D}(\bar\partial^*)$, we have that

$$|(\varphi, v)|^2 \leq \frac{1}{c}\|\varphi\|^2 \cdot \left(\int_M \|\bar\partial v_1\|^2 + \|\bar\partial^* v_1\|^2\right) = \frac{1}{c}\|\varphi\|^2 \cdot \|\bar\partial^* v_1\|^2.$$

Define the linear map T which maps $w = \bar\partial^* v$ to (v, φ). We need to check that this is well defined. Namely if $w = \bar\partial^* v_i$ for v_1 and v_2, then $(v_1, \varphi) = (v_2, \varphi)$. This can be seen as follows. Since $\bar\partial^*(v_1 - v_2) = 0$, the above estimate shows that $(v_1, \varphi) = (v_2, \varphi)$. This linear functional is defined on a closed subspace and is bounded. Hence by the Hahn-Banach and Riesz representation, there exist a ψ such that $T(w) = (w, \psi)$ with the estimate that $\|\psi\| \leq \|T\|$. This implies that for $w = \bar\partial^* v$,

$$(v, \varphi) = (w, \psi) = (\bar\partial^* v, \psi) = (v, \bar\partial\psi).$$

Since it holds for all $v \in \mathcal{A}^{p,q}$ we have that $\bar\partial\psi = \varphi$. □

In the application, a regularity result which asserts that if φ is smooth then the constructed solution ψ is smooth (with estimates) is needed. Another useful observation is that in the above argument we only need (2) holds over the support of φ. If we locally (near a point z_0) choose $s = z^i$ and get a global section by multiplying a cut-off function. Let $\varphi = \bar\partial s$. Then φ vanishes near z_0. This allows one to put a weight $e^{-\psi}$ with $\psi = C\rho(x)\log(|z|^2)$, and ρ being a cut-off function. Even though ψ has a singularity at the origin, it does not affect the estimate (2) over the support of φ. The embedding theorem can be done by solving the $\bar\partial$ with a weight of such on the metric of $\mathcal{L}$. The will make sure that the constructed sections give a mapping $\iota_{\mathcal{L}^k}$ whose differential $d\iota$ is $1-1$, hence locally an embedding. Extending the argument to two different points z_0 and z_1 (some definite distance apart) and constructing one holomorphic section which behaves as z^i near one point and behaves as 1 at the other allows us to construct holomorphic sections of $\mathcal{L}^{\otimes k}$ to separate two points. With a bit more care this construction proves Kodaira's embedding theorem, which was proved originally using *the blow-up tricks*.

The Frobenius theorem asserts that an involutive distribution Σ (a k-dimensional distribution is a k-dimensional subspace Σ_x at every point of x, which varies smoothly as $x \in M$ varies), namely $\forall X, Y \in \Sigma$, $[X, Y] \in \Sigma$, must be integrable in the sense that $\forall x$, there exits a neighborhood and a coordinate with x being the origin such that the tangent space of the subspaces $\{x \,|\, x_{k+1} = c_{k+1}; \cdots; x_n = c_n\}$ coincide with Σ everywhere in the neighborhood (in other words $\Sigma = \mathrm{span}\{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^k}\}$). One proof of this result resembles the proof of Poincaré's Lemma (see for example Taubes: *Differential Geometry* **Reading**: pages 166-169). Here we focus on two complex/holomorphic versions. One provides a criterion about when a complex vector bundle over a complex manifold can be endowed with a holomorphic structure. The second is a nonlinear version of the first giving a criterion about when an almost complex structure on an even dimensional manifold is integrable (namely induced by a complex structure). The proof of the first one is similar to the proof of Dolbeault's Lemma. The proof of the second one uses the L^2-$\bar\partial$-estimate (following Kohn-Spencer).

For the first one, let $D : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ be a connection of a complex vector bundle E. As before $D = D' + D''$ be the decomposition into $\mathcal{A}^{1,0}(E)$ and $\mathcal{A}^{0,1}(E)$. Locally write it as θ, defined by $D(e_i) = \theta_i^j e_j$. Then $\theta^{1,0}$ and $\theta^{0,1}$ are the mapping corresponding to D' and D''. In general $D'' \cdot D'' \neq 0$, and locally $D'' \cdot D''$ is given by $\bar{\partial}\theta^{0,1} + \theta^{0,1} \wedge \theta^{0,1}$, since

$$D'' \cdot D''(e_i) = D''((\theta^{0,1})_i^j e_j) = \bar{\partial}((\theta^{0,1})_i^j)e_j + e_k(\theta^{0,1})_j^k \wedge (\theta^{0,1})_i^j.$$

Since for a holomorphic vector bundle E, $D'' = \bar{\partial}$, hence $D'' \cdot D'' = 0$ (note that this holds always when $m = 1$). The converse is the following theorem of Koszul-Malgrange (1958).

Theorem 0.2 (Frobenius-1)**.** *Let E be a complex vector bundle over a complex manifold M. Assume a connection of E satisfies that $D'' \cdot D'' = 0$. Then there exists a holomorphic structure on E such that D is a connection compatible with this holomorphic structure.*

Proof. For the frame $\{e_i\}$, let $\eta = \theta^{0,1}$. If $\tilde{e}_i g_j^i = e_j$ it is easy to see that $\tilde{\eta} = g\eta g^{-1} - \bar{\partial} g \cdot g^{-1}$. If we can find g such that $\tilde{\eta} = 0$, then we call $\tilde{e}$ a holomorphic frame. It then can be checked that the transition between the holomorphic frames is holomorphic. Denote $\eta(\frac{\partial}{\partial \bar{z}^\alpha})$ as $\eta_{\bar{\alpha}}$. Hence $\eta = \eta_{\bar{\lambda}} d\bar{z}^\lambda$. The integrability condition amounts to $\bar{\partial}\eta + \eta \wedge \eta = 0$. The idea is similar to the proof of Dolbeault's lemma via an induction argument. The goal is that $\forall p$ find a small neighborhood U_p and $\{\tilde{e}_i\}$ with $\tilde{e}_i g_j^i = e_j$ such that with respect to the $\{\tilde{e}_i\}$ the connection form $\tilde{\eta} = 0$. This amounts to solving a matrix form $\bar{\partial}$ equation:

$$\tilde{\eta} = g\eta g^{-1} - \bar{\partial} g \cdot g^{-1} = 0; \text{ equivalently } \frac{\partial g}{\partial \bar{z}^\lambda} - g\eta_{\bar{\lambda}} = 0, \text{ for all } 1 \leq \lambda \leq m.$$

If by induction we have already achieved that $\eta_{\bar{\lambda}} = 0$ for $1 \leq \lambda \leq p-1$, then $\eta = \eta_{\bar{\alpha}} d\bar{z}^\alpha$ with $\alpha \geq p$. We shall show we can find g such that $\tilde{\eta}$ satisfies $\tilde{\eta}_{\bar{\lambda}} = 0$ for $1 \leq \lambda \leq p$. The integrability condition implies that $\frac{\partial}{\partial \bar{z}^\lambda}\eta_{\bar{\alpha}} = 0$ for $p \leq \alpha \leq m$. The key observation is that if g, *the gauge transformation* is holomorphic in $z_1, \cdots, z_{p-1}$, $\tilde{\eta}_{\bar{\lambda}} = 0$ for $1 \leq \lambda \leq p-1$. Then the problem is reduced to solving $\frac{\partial g}{\partial \bar{z}^p} = g\eta_{\bar{p}}$ and showing that the solution g depends holomorphically on $\eta_{\bar{p}}$ (which is assume to holomorphically in z^λ) involved. For $\lambda = p$, we write $\rho = \eta_{\bar{p}}, w = z_p$. Then we need to solve $\frac{\partial}{\partial \bar{w}} g = g\rho$. Here we assume that $\frac{\partial \rho}{\partial \bar{z}^\beta} = 0$ for $1 \leq \beta \leq p-1$. Since we only need to solve the problem locally we may assume that ρ has a compact support. Since $\eta = \rho(w)dw$, after the scaling $z \to w = rz$, become $[\rho(rz)r]dz$, we may assume that ρ is small if we are willing to shrink the ball involved in the local trivialization. The way of the construction is to write $g = I + f$ with I being the identity, and construct f. Let $L_\rho(f)$ be the operator defined as

$$L_\rho(f)(w) = \int_{\mathbb{C}} \frac{(I+f)\rho}{\xi - z} \left(\frac{1}{2\pi\sqrt{-1}} d\xi \wedge d\bar{\xi} \right).$$

The formula is motivated by the proof of Dolbeault's Lemma. It is easy to see that

$$\|L_\rho(f_1) - L_\rho(f_2)\|_\infty = \left\| \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{C}} \frac{(f_1 - f_2)\rho}{\xi - w} d\xi \wedge d\bar{\xi} \right\|_\infty \leq \delta \|f_1 - f_2\|_\infty$$

with $\delta < 1$ if $r << 1$. By the Contraction Mapping Theorem we can find a fixed point f,

$$f = \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{C}} \frac{(I+f)\rho}{\xi - w} \left(d\xi \wedge d\bar{\xi} \right).$$

Let $g = I + f$. The fixed point can be found with $\|f\|_\infty < \frac{1}{2}$ to ensure that g is invertible. It is easy to see $\frac{\partial}{\partial \bar{z}} g = g\rho$. The induction completes the proof. ☐

The second Frobenius theorem involves a concept called *the almost complex structure* on the tangent bundle, which is a bundle map $J : TM \to TM$ satisfying that $J^2 = -\operatorname{id}$. Extending it to $T_{\mathbb{C}}M$ and decomposes $T_{\mathbb{C}}M = T'M \oplus T''M$ point-wisely according to eigenvalue being $\sqrt{-1}$ and $-\sqrt{-1}$, see Kobayashi-Nomizu Ch9). *Nijenhuis tensor* is defined as:

$$N(X,Y) = [X,Y] - [JX,JY] + J[JX,Y] + J[X,JY], \forall X,Y \in T_xM.$$

Note that $N(X,JX) = 0$ always (compare with $(D'')^2 = 0$ for $m = 1$). It can be checked easily that $Z = X - \sqrt{-1}JX$ is a $(1,0)$ type vector, namely $JZ = \sqrt{-1}Z$. If the manifold is complex and J is induced by multiplying $\sqrt{-1}$, then $[Z,W]$ is of type $(1,0)$, if Z,W are of type $(1,0)$. Hence $J[Z,W] = \sqrt{-1}[Z,W]$. Now letting $W = Y - \sqrt{-1}JY$, a direct calculation shows that this holds if and only if $N(X,Y) = 0$ (such J is called *integrable*). Namely the integrability is a necessary condition of J being induced by a complex manifold structure. It is a theorem of Newlander-Nirenberg (1957) (**Reading**: Ann. of Math. 1957) asserting that $N = 0$ is also sufficient.

Theorem 0.3 (Frobenius-2)**.** *Let M be a smooth manifold such that it admits an almost complex structure J on $T_{\mathbb{C}}M$ with $N = 0$. Then there exists a holomorphic structure on M which is compatible with J.*

The complex structure J naturally acts on $T^*_{\mathbb{C}}M$ and decomposes it into direct sums of $(T'M)^*$ and $(T''M)^*$ as eigenspaces of eigenvalue $\sqrt{-1}$ and $-\sqrt{-1}$ respectively. Locally we may choose $\{\omega^i\}_{1\le i\le m}$ such that $\{\omega^i\}$ forms a basis of $(T'M)^*$ and $\{\bar\omega^i\}$ forms a basis of $(T''M)^*$. If $\{e_i\}$ and $\{\bar e_i\}$ are the dual basis then we write $df = e_i(f)\omega^i + \bar e_i(f)\bar\omega_i$. We call the first sum $(1,0)$-type, and the second part $(0,1)$ type. From this we can define the (p,q) type differential forms. Now we can also reformulate the condition $N = 0$ as that for any $(0,1)$ type 1-form u, du has no $(2,0)$ type component. This can be seen as for Z,W of $(1,0)$-type vectors, u is a $(0,1)$-type 1-form,

$$du(Z,W) = Z(u(W)) - W(u(Z)) - u([Z,W]) = -u([Z,W]) = 0, \iff [Z,W] \in T'M.$$

Under this assumption, it is easy to see that for u, a $(1,0)$-type 1-form, du has no $(0,2)$-components. This immediately implies that for a (p,q)-form u, du only has $(p+1,q)$ and $(p,q+1)$ component. Then we can define ∂ and $\bar\partial$ to be the $(p+1,q)$ and $(p,q+1)$ type components. Hence from $d^2 = 0$, follows that $\partial^2 = \bar\partial^2 = \partial\bar\partial + \bar\partial\partial = 0$.

Proof. The proof amounts to finding functions $U^1, \cdots, U^m$ locally such that $\bar\partial U^i = 0$ and $\{dU^i\}$ linearly independent locally. First for $\{\omega^i\}$ which is a basis of $(T'M)^*$ in a small neighborhood of z_0 it is easy to find u^i such that $du^j = \omega^j$ at z_0. We shall modify u^j to achieve our goal. The key here is the L^2-estimate with the weight given by a convex increasing function $\chi(t)$ composed with $\psi = \sum_{i=1}^n |x^i|^2$, where $(x^1, \cdots, x^n)$ are local coordinate centered a given point z_0. It is easy to see that ψ is convex locally. Evoke the transformation $\pi_r : z \to w = rz$ as before, by the construction of u^j, since $(du^j)(rz) \to \omega^j(0)$ as $r \to 0$, we have that

$$du^j - \frac{\pi_r^*\omega^j}{r} = e_k(u^j)\omega^k + \bar e_k(u^j)\bar\omega^k - \frac{\pi_r^*\omega^j}{r} \to 0, \text{ in particular } \bar\partial u^j \to 0.$$

The L^2-estimate (3) together with the regularity estimate implies that there exists a solution v^j_r with $\bar\partial v^j_r = \bar\partial u^j$ such that $(\|v^j_r\|_\infty + \|D^\alpha v^j_r\|_\infty) = O(r)$ as $r \to 0$ with $|\alpha| = 1$. This estimate makes sure that v^j differs from u^j. Moreover $dv^j_r \to 0$ as $r \to 0$. Let $U^j = u^j - v^j_r$ for $r << 1$. Direct checking shows that $\{U^j\}$ provides a local coordinate. □

Lecture 7 –Uniqueness of $\mathbb{P}^m$ and the Frankel's Conjecture **by L. Ni**

Here we first prove a theorem of Hirzebruch-Kodaira: *If M^m is a compact Kähler manifold which is diffeomorphic to $\mathbb{P}^m$. Then it is biholomorphic to $\mathbb{P}^m$.* In fact what proved by them is a bit weaker than this when m is even, since they assumed additionally that $c_1(M) \neq -(m+1)g$ with g being a generator of $H^2(M, \mathbb{Z})$. This was later removed via a result of Aubin-Yau on the existence of Kähler-Einstein metric (for $c_1(M) < 0$). Ochia-Kobayashi then proved a cohomological criterion. Built upon this criterion Siu-Yau (1980) then proved the Frankel's conjecture: *Any compact Kähler manifold (M, g) with G-positive curvature must be $\mathbb{P}^m$.* This is weaker than Mori's theorem (proved in 1979, without assuming the Kählerity). The key new ideas and techniques in Mori's proof is to construct rational curves and their deformations. Here we present a proof of Siu-Yau of the Frankel's conjecture.

Theorem 0.1 (Kobayashi-Ochia)**.** *Let M^m be a compact complex manifold with an ample line bundle F. If $c_1(M) \geq (m+1)c_1(F)$, then M is biholomorphic to $\mathbb{P}^m$. If $c_1(M) = mc_1(F)$, then M is biholomorphic to a hyperquadric in $\mathbb{P}^{m+1}$.*

Even though Kobayashi-Ochia is after Hirzebruch-Kodaira, the proof of the above theorem is mostly algebraic, hence here we shall reduce Hirzebruch-Kodaira to Kobayashi-Ochia's result and omit the proof of Theorem 0.1 (**Reading:** *J. Math. Kyoto Univ.*, 1973). Basic topology tells us that $H^2(\mathbb{P}^m, \mathbb{Z}) = \mathbb{Z} = H^2(M, \mathbb{Z})$ and the only nontrivial cohomologies are H^{2k}. We may pick a generator $g \in H^2(\mathbb{P}^m, \mathbb{Z})$ such that $g^m([M]) = 1$. Since g must be a multiple of the Kähler class we may assume that $[\omega] = g$ by picking a different Kähler metric. With this choice we have that $\int_M \omega^m = 1$. Similarly $[c_1(M)]$ (in this lecture we also use $c_1(M)$ to denote the class when the context is clear) is a multiple of g. Since $[c_1(M)]$ is also an integral class, there exists an integer λ such that $[c_1(M)] = \lambda g$. For the theorem it suffices to prove $\lambda = (m+1)$. Below we shall prove this. First use $b_k = \sum_{p+q=k} h^{p,q}$ we can conclude that $h^{p,o} = 0 = h^{0,p}$. This shows that

$$1 = \sum_{q=0}^{m} (-1)^q h^{0,q} = \chi(M) = \int_M e^{\frac{c_1(M)}{2}} \left(\frac{\frac{g}{2}}{\mathrm{sh}(\frac{g}{2})} \right)^{m+1}. \tag{1}$$

The last equation is via the Riemann-Roch Theorem and the consideration via Pontrjagin classes, which are diffeomorphic invariants. The point is that $\left(\frac{\frac{g}{2}}{\mathrm{sh}(\frac{g}{2})} \right)^{m+1}$ can be written in terms of Pontrjagin classes. By consider the coefficient morphism $\mathbb{Z} \to \mathbb{Z}_2$ we also have that $\lambda \equiv (m+1) \mod (2)$. Hence $\lambda = 2s + (m+1)$ for some integer s. In

$$e^{\frac{c_1(M)}{2}} \left(\frac{\frac{g}{2}}{\mathrm{sh}(\frac{g}{2})} \right)^{m+1} = e^{sg + \frac{m+1}{2} g} \left(\frac{\frac{g}{2}}{\mathrm{sh}(\frac{g}{2})} \right)^{m+1} = e^{sg} \left(\frac{g}{1 - e^{-g}} \right)^{m+1}$$

the coefficient of g^m is the same as the dimension of $H^0(\mathbb{P}^m, \mathcal{O}(s))$ since that is what the Riemann-Roch and the vanishing theorem provide for $\mathbb{P}^m$ and M. But $\dim(H^0(\mathbb{P}^m, \mathcal{O}(s))) = \binom{m+s}{m}$ (See Griffiths-Harris). Thus if $s \geq 0$, $1 = \binom{m+s}{m}$, which implies $s = 0$. Or $s < 0$, then $1 = (-1)^m h^{0,m}(\mathcal{O}(k)) = (-1)^m \binom{-s-1}{m}$, which implies $s = -(m+1)$ and m is even. The possibility that $s = -(m+1)g$ (hence $\lambda = -(m+1)g$) is ruled out by a result of Aubin-Yau. If $c_1(M) = -(m+1)g$ with g being the Kähler class. Aubin-Yau's theorem then asserts that one can find a a metric within the same Kähler class such that $R_{\alpha\bar\beta} = -(m+1)g_{\alpha\bar\beta}$. A calculation of Chen-Ogiue (which traces further back to Berger and Lascoux) then implies,

$$(2) \qquad (-1)^m c_1^m \le (-1)^m \frac{2(m+1)}{m} c_1^{m-2}\cdot c_2;\ \text{or}\ c_1^2\wedge\omega^{m-2} \le \frac{2(m+1)}{m} c_2\wedge \omega^{m-2},$$

with the equality holds if and only if M is a compact quotient of $\mathbb{D}^m$. Notice that $c_1^2 = (m+1)^2 g^2$, which is topological and the Pontryagin class $p_1 = c_2 - c_1^2$ is topological. Hence the above $(\frac{2(m+1)}{m}c_2 - c_1^2)\wedge g^{m-2}$ is topological. Thus it is the same as that of $\mathbb{P}^m$. However on $\mathbb{P}^m$ (2) holds as an equality by direct checking. It implies that equality in (2) holds on M. Then M is a compact quotient $\mathbb{D}^m$, which is not simply-connected. This is a contradiction! Hence $\lambda = -(m+1)$ does not occur. From the proof one can see the Kähler condition was used from the beginning to assume that c_1 is a multiple of the generator (which taken as the Kähler class).

Despite that Siu-Yau's result is a special case of Mori's earlier result. Its proof however has the feature that it uses tools from differential geometry only, except a simple lemma (**Reading**: Huybrechts, pages 244-245) due to Grothendieck on vector bundles over $\mathbb{P}^1$ (the vector bundles over $\mathbb{P}^m$ is still a subject under active studies).

Lemma 0.1 (Grothendieck)**.** *Every holomorphic vector bundle E over $\mathbb{P}^1$ splits into direct sum of line bundles $E = \oplus\mathcal{O}(a_i)$. The ordered sequence $a_1 \ge a_2 \ge \cdots \ge a_r$ is uniquely determined.*

Below is a proof of Siu-Yau's result. If repeated indices appear in the summation the Einstein's convention is used. Here $1 \le \alpha, \beta, \gamma, \delta \le m$ and $1 \le i, j, k, l \le n$ with $m = \dim M$ and $n = \dim N$. We use a stability inequality obtained by complexifying the Eells-Sampson's second variation formula.

For $u : M \to N$ a harmonic map between two Riemannian manifolds, if $V \in \Gamma(E)$ with $E = u^*TN$ is a variational vector fields (namely there exists a family of variation $u(x, s)$ such that $\frac{\partial}{\partial s}u\big|_{s=0} = V$) with compact support, then

$$(3) \qquad \frac{\partial^2}{\partial s^2}\bigg|_{s=0} \mathcal{E}(u(\cdot, s)) = \int_M |\nabla_{e_\alpha} V|^2 - R(V, du(e_\alpha), du(e_\alpha), V)$$

where $\{e_\alpha\}$ is a local orthonormal frame of T_xM, $\mathcal{E}(u) = \frac{1}{2}\int_M e(u)$ is the energy of the map. This was proved first in Eells-Sampson's 1964 paper (**Reading**: *Amer. J. Math.*, Ch II, equation (3)). Polarizing the right hand side we have the index form

$$I(V, W) = \int_M \langle\nabla_{e_\alpha} V, \nabla_{e_\alpha} W\rangle - R(V, du(e_\alpha), du(e_\alpha), W).$$

The Jacobi operator is the Euler-Lagrange equation, a second order elliptic linear operator, associated with $I(V, W)$.

For applications it is useful to complexify TN (at the same time also complex extends R the curvature tensor) and $I(\cdot,\cdot)$. This particularly make sense in case that M is a complex/Kähler manifold. If (M^m, g) is a Kähler manifold of complex dimension m (real dimension $2m$) let $\{E_\alpha = \frac{1}{\sqrt{2}}(e_\alpha - \sqrt{-1}Je_\alpha)\}$ be the unitary frame of $T'M$ associated with $\{e_1, \cdots, e_m, Je_1, \cdots, Je_m\}$. Direct calculation together with the first Bianchi identity gives the following proposition.

Proposition 0.1. *Let (M^m, g) be a Kähler manifold. For $V, W \in \Gamma(u^*T_{\mathbb{C}}N)$ with compact support, the complex bilinear extension of $I(\cdot,\cdot)$ satisfies*

$$I(V,\overline{W}) = 2\int_M \left(\langle \nabla_{\overline{E}_\alpha} V, \nabla_{E_\alpha}\overline{W}\rangle - \langle R_{V,du(E_\alpha)}du(\overline{E}_\alpha), \overline{W}\rangle\right) d\mu. \tag{4}$$

Proof. Direct calculation shows that

$$\begin{aligned}\int_M \langle \nabla_{\overline{E}_\alpha} V, \nabla_{E_\alpha}\overline{W}\rangle &= \frac{1}{2}\int_M \langle \nabla_{e_\alpha} V, \nabla_{e_\alpha}\overline{W}\rangle + \langle \nabla_{Je_\alpha} V, \nabla_{Je_\alpha}\overline{W}\rangle \\ &+\frac{\sqrt{-1}}{2}\int_M \langle \nabla_{Je_\alpha} V, \nabla_{e_\alpha}\overline{W}\rangle - \langle \nabla_{e_\alpha} V, \nabla_{Je_\alpha}\overline{W}\rangle;\end{aligned}$$

$$\begin{aligned}\int_M \langle R(V, du(E_\alpha))du(\overline{E}_\alpha), \overline{W}\rangle &= \frac{1}{2}\int_M \langle R(V, du(e_\alpha))du(e_\alpha), \overline{W}\rangle \\ &+\frac{1}{2}\int_M \langle R(V, du(Je_\alpha))du(Je_\alpha), \overline{W}\rangle \\ &-\frac{\sqrt{-1}}{2}\int_M \langle R(V, du(Je_\alpha))du(e_\alpha), \overline{W}\rangle \\ &+\frac{\sqrt{-1}}{2}\int_M \langle R(V, du(e_\alpha))du(Je_\alpha), \overline{W}\rangle.\end{aligned}$$

The terms with $\sqrt{-1}$ factor get cancelled out due to integration by parts and the first Bianchi identity. □

Hence in terms of complex notations, the Jacobi operator $\mathcal{J}$ can be expressed as

$$\mathcal{J}(V) = -\nabla^2_{E_\alpha,\overline{E}_\alpha} V - R_{V,du(E_\alpha)}du(\overline{E}_\alpha).$$

If (N, h) is also a Kähler manifold, $du : T_{\mathbb{C}}M = T'M \oplus T''M \to T_{\mathbb{C}}N = T'N \oplus T''N$ splits into maps $\partial u : T'M \to T'N$, $\bar{\partial} u : T'M \to T''N$ and their conjugates. If in terms of the holomorphic coordinate $(w^1, \cdots, w^n)$ of N we write $V = u^i_{\bar{s}}\frac{\partial}{\partial w^i} + u^{\bar{i}}_{\bar{s}}\frac{\partial}{\partial \overline{w^i}}$ noting that R vanishes if first or last two entries are taking values both $T'N$ (or both in $T''N$), taking $W = V$ (4) yields a formula similar to the second variation formula of Siu-Yau (page 192)

$$\begin{aligned}I(V,\overline{V}) &= \int_M g^{\alpha\bar{\beta}} u^i_{\bar{s}\bar{\beta}}\overline{u^j_{\bar{s}\bar{\alpha}}}h_{i\bar{j}} + \int_M g^{\alpha\bar{\beta}} u^i_{s\alpha}\overline{u^j_{s\beta}}h_{i\bar{j}} \\ &- \int_M g^{\alpha\bar{\beta}} R_{i\bar{j}k\bar{l}} u^i_{\bar{\beta}} u^{\bar{j}}_{\alpha} u^k_{\bar{s}} u^{\bar{l}}_{s} - \int_M g^{\alpha\bar{\beta}} R_{i\bar{j}k\bar{l}} u^i_{\alpha} u^{\bar{j}}_{\bar{\beta}} u^k_{s} u^{\bar{l}}_{\bar{s}} \\ &+ 2\mathcal{R}e\int_M g^{\alpha\bar{\beta}} R_{i\bar{j}k\bar{l}} u^i_{\alpha} u^k_{\bar{\beta}} u^{\bar{j}}_{s} u^{\bar{l}}_{\bar{s}}.\end{aligned} \tag{5}$$

We shall use this to show the holomorphicity of a stable harmonic map $u : \mathbb{S}^2 \to N$ under the assumption that N has positive bisectional curvature (in fact the orthogonal bisectional curvature $B^{\perp} > 0$ would be sufficient). The key is to construct holomorphic variations via the above Grothendieck Lemma and Riemann-Roch theorem. The first complex Frobenius theorem of last lecture can be applied here to endow a holomorphic structure on $E =$

$u^{-1}T_{\mathbb{C}}N = u^{-1}T'N \oplus u^{-1}T''N = E_1 \oplus E_2$. By G-Lemma E then splits into the sum of holomorphic line bundles:

$$E_1 = L_1^1 \oplus \cdots \oplus L_n^1; \quad E_2 = L_1^2 \oplus \cdots \oplus L_n^2.$$

Now rank them according to their first Chern class with

$$c_1(L_1^1) \geq \cdots \geq c_1(L_n^1); \quad c_1(L_1^2) \geq \cdots \geq c_1(L_n^2).$$

Note that $c_1(E) = 0$ and E_i are conjugate each other, we may arrange L_j^i so that $(L_j^1)^* = L_{n-j+1}^2$. Without the loss of generality we assume $c_1(L_1^1) \geq 0$, hence it admits a nonzero holomorphic section η, which can be expressed as $\eta^i \frac{\partial}{\partial w^i}$. We choose the deformation such that $\partial u(\frac{\partial}{\partial \bar{s}}) = \eta^i \frac{\partial}{\partial w^i}$ and $\partial u(\frac{\partial}{\partial s}) = 0$. Using this variational vector field, (5) implies that $\partial u(\frac{\partial}{\partial \bar{z}}) = 0$ since

$$0 \leq I(V, \overline{V}) = -\int_{\mathbb{S}^2} g_z^{-1} R \left(\partial u \left(\frac{\partial}{\partial \bar{z}} \right), \overline{\partial u \left(\frac{\partial}{\partial \bar{z}} \right)}, \eta, \overline{\eta} \right) d\mu_{\mathbb{C}}$$

In the case when only orthogonal bisectional curvature is positive ($B^\perp > 0$), note that $\langle \eta, \overline{\frac{\partial u}{\partial \bar{z}}} \rangle = \langle \eta, \frac{\partial u}{\partial z} \rangle = 0$ due to the finite energy of u and the harmonic map equation.

To finish the proof, first observe that the positivity of the bisectional curvature implies that $b_2(N) = 1$ (also true if $B^\perp > 0$). Hence $H^2(N, \mathbb{Z}) = \mathbb{Z} \oplus \mathrm{Tor}(H_1(N, \mathbb{Z}))$. Since the manifold is simply-connected (which is not true in the case $B^\perp > 0$ only) we have that $H^2(N, \mathbb{Z}) = \mathbb{Z}$. As in the previous case, this implies that there exists a line bundle F, $c_1(F)$ generates $H^2(N, \mathbb{Z})$. Let g be the element in the free part of $H_2(N, \mathbb{Z})$ such that $\langle c_1(F), g \rangle = 1$. Since $H_2(N, \mathbb{Z}) \simeq \pi_2(N)$, g can be represented by stable harmonic spheres via the existence result of Sacks-Uhlenbeck below. Here for simplicity we assume that g is given by a stable harmonic sphere (in general, Sacks-Uhlenbeck can only express g as sums of stable harmonic spheres). The above discussion shows that $u : \mathbb{S}^2 \to N$ is a rational curve. Now $u^{-1}T'N$ is a holomorphic bundle over $\mathbb{S}^2$ with $T'\mathbb{P}^1$ as a sub-bundle. Hence

$$c_1(u^{-1}T'N) = c_1(T'\mathbb{P}^1) + c_1([D]) + \sum_{i=2}^{n} c_1(Q_i)$$

with Q_i be the summand of the quotient bundle, D being the degeneracy divisor of du. Since $c_1(T'\mathbb{P}^1) = 2$ and $c_1(Q_i) > 0$ (due to that the curvature is positive and taking quotient does not decrease the curvature), we have that $c_1(u^{-1}T'N) \geq n + 1$. This shows that $c_1(N) \geq (n+1)F$. Hence by Kobayashi-Ochia we have the claimed result. The general case requires a deformation argument to ensure the assumption that g is represented by a stable harmonic sphere.

We state the general existence result of Sacks-Uhlenbeck below.

Theorem 0.2 (Sacks-Uhlenbeck)**.** *Assume that* $\dim(M) = 2$ *and* M *is oriented. (i) If* $\pi_2(N) = 0$*, then every homotopy class of maps in* $[M, N]$ *contains a a minimizing harmonic map; (ii) If* $\pi_2(N) \neq 0$*, then there exists a set of free homotopy class* Λ_i *(each class* Λ_i *can be identified with an orbit of certain element* $g \in \pi_2(N)$ *acted by* $\pi_1(N)$*) such that* $\{\lambda \in \Lambda_i\}$ *generates* $\pi_2(N)/\pi_1(N)$ *(which can be identified with* $[\mathbb{S}^2, N]$*), and each* Λ_i *contains a conformal branched minimal immersion of* $\mathbb{S}^2$ *which minimizes the energy and the area in its homotopy class.*