Kodaira embedding and a Schwarz Lemma after Yau-Royden

Lei Ni

University of California, San Diego

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- A Kähler manifold (M, g) is a complex one whose Kähler form $\omega_g = \frac{\sqrt{-1}}{2} g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$ is d-closed.
- Kodaira (1954): Assuming the Kähler form is integral, a compact Kähler manifold can be holomorphically embedded into the complex projective $\mathbb{P}^m$ (called projective).

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- Hence ‘projectivity’=‘being algebraic’; and the embedding avails us the algebraic tools to study some Kähler manifolds.
- ’Forgetting’ allows PDEs/geometric method to study the algebraic manifolds, e.g. yielding the Hodge theorem/structure, Rieman-Roch-Hirzebruch index theorem etc.

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- Given a Kähler manifold (M, g), when it admits a positive line bundle?
- The 'canonical choice' is the canonical line bundle $(K_M, \det(g)^{-1})$ or anti-canonical line $K_M^{-1} = \det T'M$ bundle.

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- Mok: If a Kähler manifold (M, g) satisfies $B \geq 0$, $b_2 = 1$, then M is a compact HSS.

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- More importantly there are many tori of complex dimension 2 which is not projective.
- Our first result/curvature captures, to some degree, the essential connection between the intrinsic positivity and the projectivity.

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Clearly for $k = 1$, $S_1(x) \geq \lambda$ is the same as $H(X) \geq \lambda |X|^4$, and $S_m(x) = S(x)$.

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  - By Kodaira-Spencer, there exists smooth deformations of Kähler metrics among a family of holomorphic deformations. Hence the result also implies the stability of the projectivity for such manifolds.

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- First ingredient: a $\partial\bar{\partial}$-lemma.

$$\partial\overline{\partial}|s|^2 = \langle \nabla s, \overline{\nabla s} \rangle - \widetilde{R}(s, \bar{s}, \cdot, \cdot)$$

where $\widetilde{R}$ stands for the curvature of the Hermitian bundle $\bigwedge^p \Omega$, and $\Omega = (T'M)^*$ is the holomorphic cotangent bundle of M. The metric on $\bigwedge^p \Omega$ is derived from the metric of M^m.

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- Second ingredient: 2nd variation consideration (on the minimal 2-subspaces, or k-spaces) is the key (this is motivated by Wiliking's proof of invariant conditions for Ricci flow). To prove the theorem we apply the maximum principle at x_0, where $|s|^2$ attains its maximum (s being a holomorphic $(2, 0)$-form).

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- In view of the compactness of the Grassmannians we find a complex two plane Σ in $T'_{x_0}M$ such that $S_2(x_0, \Sigma) = \inf_{\Sigma'} S_2(x_0, \Sigma') > 0$.

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\end{aligned}$$

- The novelty of our proof is to use the minimality of Σ to get useful estimates first, and then apply them to tracing $\partial\bar{\partial}$-Lemma over Σ (only). B. Andrews (then adapted by Brendle and others) applied a similar trick to the diagonal manifolds (e.g. half of the dimension of the product manifold).

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- Define $Ric(x, \Sigma)$ as the Ricci curvature of the curvature tensor restricted to the k-dimensional subspace $\Sigma \subset T'_x M$. Namely $Ric(x, \Sigma)(v, \bar{v}) = \sum_{i=1}^{k} R(E_i, \overline{E}_i, v, \bar{v})$ with $\{E_i\}$ being a unitary basis of Σ. We say that $Ric_k(x) > 0$ if $Ric(x, \Sigma) > 0$ for every k-dimensional subspace Σ.

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- Theorem (Ni, 2019)

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- The proof uses 1) the above estimates and considerations, 2) Applying maximum principle to the co-mass of the forms.

D1. Kobayashi hyperbolicity and the Schwarz Lemma of Yau-Royden

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- Schwarz Lemma (Yau-Royden): Let $f : \Sigma \to N^n$ be a holomorphic map. Assume that the holomorphic sectional curvature of N, $H(Y) \leq -\kappa |Y|^4$ and the curvature of Riemann surface Σ, $Ric(X, \overline{X}) \geq -K|X|^2$ with $\kappa, K > 0$. Then
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$$\|\partial f\|^2 \leq \frac{K}{\kappa}.$$
- It generalizes the Ahlfors's result and implies the 1-hyperbolicity of N, if N is compact and $H^N(Y) < 0$. Hence $H < 0$ is a very restrictive condition. Conditions $S_k < 0$, and $Ric_k < 0$ for $k \geq 2$ are more flexible.

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- Theorem (N-2018)

Let (M, g) be a complete Kähler manifold such that the holomorphic sectional curvature $H^M(X)/|X|^4 \geq -K$, and (N^n, h) be a Kähler manifold with $H^N(Y) < -\kappa |Y|^4$ for some $\kappa > 0$. Let $f : M \to N$ be a holomorphic map. Then

$$\|\partial f\|_m^2 \leq \frac{K}{\kappa}, \tag{0.1}$$

provided that the bisectional curvature of M is bounded from

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- The proof uses a viscosity consideration from PDE theory. The key is to construct a smooth barrier.

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- Corollary: *the equivalence of the negativities of the holomorphic sectional curvature implies the equivalence of the metrics.*
- If two Kähler metrics g_1 and g_2 satisfy that

$$-L_1|X|^4_{g_1} \le H_{g_1}(X) \le -U_1|X|^4_{g_1}, \quad -L_2|X|^4_{g_2} \le H_{g_2}(X) \le -U_2|X|^4_{g_2}$$

then for any $v \in T'_x M$ we have the estimates:

$$|v|^2_{g_2} \le \frac{L_1}{U_2}|v|^2_{g_1}; \quad |v|^2_{g_1} \le \frac{L_2}{U_1}|v|^2_{g_2}.$$

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(i) Assume that $\dim_{\mathbb{C}} M = m \leq n = \dim_{\mathbb{C}} N$. Let (M, g) be a compact Kähler manifold such that $Ric^M \geq 0$. Let (N^n, h) be a complete Kähler manifold such that $S_m^N(y) < 0$. Then any holomorphic map $f : M \to N$ must be degenerate. If the condition is relaxed to allow $S_m^N \leq 0$ then either the map is degenerate or it must be totally geodesic.
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  - For (ii) the equal dimensional case was known (Mok-Yau). Here $Ric_k^N < 0$ is a stronger assumption than $S_k < 0$.

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- Similar question can be asked for $Ric_k < 0$ for $k \in (1, n)$, which is stronger than $S_k < 0$.
- (Motivated by my metric stability result) Is a compact Kähler manifold with H close to -1 biholomorphic to a quotient of complex hyperbolic space? (Negative for Riemannian case by Gromov-Thurston. But true for the positive case due to Mori, Siu-Yau's result and the curvature pinching).

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- Quadratic orthogonal bisectional curvature (QB) is defined as $\langle R, A^2 \bar{\wedge} id - A \bar{\wedge} A \rangle$ for any Hermitian symmetric tensor A. Locally QB$> 0 = \sum_{i,j} R_{i\bar{i}j\bar{j}}(a_i - a_j)^2 > 0$, for any unitary frame $\{e_i\}$, $\vec{a} \neq c\vec{1}$.

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- A step back: $Ric^{\perp}(X, \bar{X}) \doteqdot Ric(X, \bar{X}) - H(X)/|X|^2$. $(QB) > 0$ implies $Ric^{\perp} > 0$. Wang-Zheng-N: Classical C-spaces with $b_2 = 1$ satisfy $Ric^{\perp} > 0$ (unlike $QB > 0$); A Frankel type result holds; A complete classification for $\dim_{\mathbb{C}}(M) = 3$.

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- A step further (motivated by Calabi-Vesentini's work): On a Kähler manifold (M^n, g), let $T'M$ and $T''M$ be the holomorphic and anti-holomorphic tangent bundle of M, then CQB is a Hermitian quadratic form on linear maps $A: T''M \to T'M$:

$$\mathrm{CQB}(A) = \sum_{\alpha,\beta=1}^{n} R(A(\overline{E}_\alpha), \overline{A(\overline{E}_\alpha)}, E_\beta, \overline{E}_\beta) - R(E_\alpha, \overline{E}_\beta, A(\overline{E}_\alpha), \overline{A(\overline{E}_\beta)}$$

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  where R is the curvature tensor of M and $\{E_\alpha\}$ is a unitary frame of $T'M$. ^{d}CQB can be defined as well.
- $\mathrm{CQB}_1 > 0$ implies $Ric > 0$ and $Ric^{\perp} > 0$. The local rigidity of manifolds with ${}^d\mathrm{CQB} > 0$, and that all classical C-spaces with $b_2 = 1$ satisfies $\mathrm{CQB} > 0$, ${}^d\mathrm{CQB} > 0$ (Ni-2019).

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  There are Riemannian analogues of CQB and ^{d}CQB. The result for real cases generalizes an earlier result of Böhm-Wilking on manifolds with $K \geq 0$.